LECTURE

14

Chapter
6.1 – 6.4

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Third Edition

BEAMS: SHEARING STRESS

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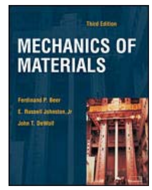

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ENES 220 – Mechanics of Materials

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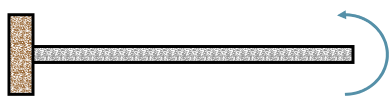
Shearing Stress in Beams

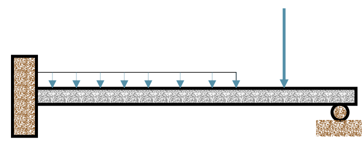
- Shear and Bending
  - Although it has been convenient to restrict the analysis of beams to pure bending, this type of loading is rarely encountered in an actual engineering problem.
  - It is much common for the resultant internal forces to consist of a bending moment and shear force.

Shearing Stress in Beams

- Shear and Bending

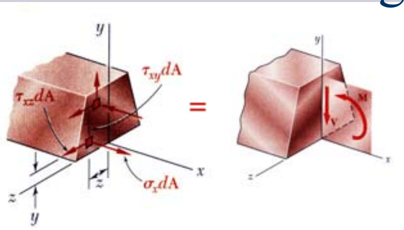


Pure Bending



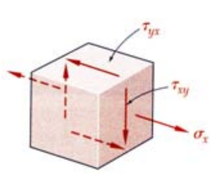
Bending and Shear Force

Shear and Bending



- Transverse loading applied to a beam results in normal and shearing stresses in transverse sections.
- Distribution of normal and shearing stresses satisfies

$$F_x = \int \sigma_x dA = 0 \qquad M_x = \int \left(y\tau_{xz} - z\tau_{xy}\right) dA = 0$$
$$F_y = \int \tau_{xy} dA = -V \qquad M_y = \int z\sigma_x dA = 0$$
$$F_z = \int \tau_{xz} dA = 0 \qquad M_z = \int \left(-y\sigma_x\right) = 0$$



- When shearing stresses are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces
- Longitudinal shearing stresses must exist in any member subjected to transverse loading.

Shearing Stress in Beams

- Shear and Bending
  - The presence of a shear force indicates a variable bending moment in the beam.
  - The relationship between the shear force and the change in bending moment is given by

$$V = \frac{dM}{dx} \qquad (42)$$

Shearing Stress in Beams

- Shear and Bending
  - Strictly speaking, the presence of shear force and resulting shear stresses and shear deformation would invalidate some of our assumption in regard to geometry of the the deformation and the resulting axial strain distribution.
  - Plane sections would no longer remain plane after bending, and the geometry

Shearing Stress in Beams

- Shear and Bending

  of the actual deformation would become considerably more involved.

  – Fortunately, for a beam whose length is large in comparison with the dimensions of the cross section, the deformation effect of the shear force is *relatively small*; and it is assumed that the longitudinal axial strains are still distributed as in pure bending.

Shearing Stress in Beams

- Shear and Bending

  – When this assumption is made, the load stress relationships developed previously are considered valid.

  – The question now being asked:

  When are the shearing effects so large that they cannot be ignored as a design consideration?

Shearing Stress in Beams

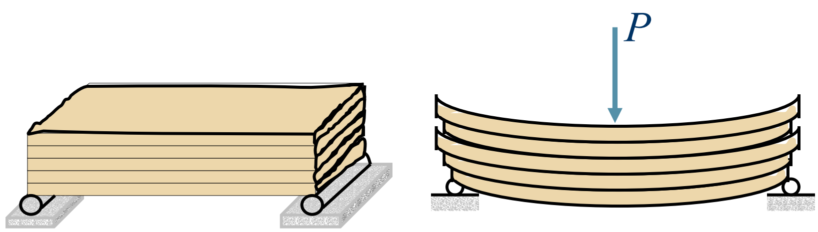
- Shear and Bending
  - It is somehow difficult to answer this question.
  - Probably the best way to begin answering this question is to try to approximate the shear stresses on the cross section of the beam.

Shearing Stress in Beams

- Shearing Stress due to Bending
  - Suppose that a beam is constructed by stacking several slabs or planks on top of another without fastening them together.
  - Also suppose this beam is loaded in a direction normal to the surface of these slabs.

Shearing Stress in Beams

- Shearing Stress due to Bending



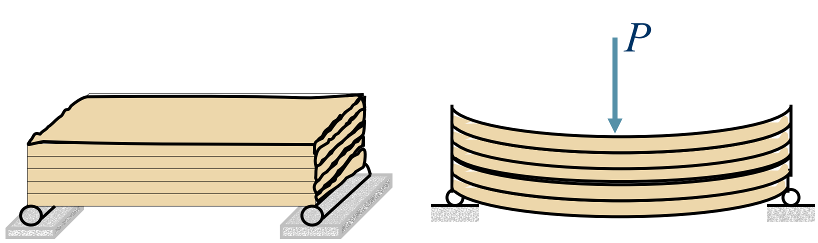
(a) Unloaded Stack of Slabs

(b) Unglued Slabs loaded

Figure 22

Shearing Stress in Beams

- Shearing Stress due to Bending



(c) Glued Slabs Unloaded

(d) Glued Slabs loaded

Figure 22 (cont'd)

Shearing Stress in Beams

- Shearing Stress due to Bending
  - When a bending load is applied, the stack will deform as shown in Fig. 22a.
  - Since the slabs were free to slide on one one another, the ends do not remain even but staggered.
  - Each of the slabs behaves as independent beam, and the total resistance to bending of *n* slabs is approximately *n* times the resistance of one slab alone.

Shearing Stress in Beams

- Shearing Stress
  - If the slabs of Fig. 22b is fastened or glued, then the staggering or relative longitudinal movement of slabs would disappear under the action of the force. However, shear forced will develop between the slabs.
  - In this case, the stack of slabs will act as a solid beam.
  - The fact that this solid beam does not

Shearing Stress in Beams

- Shearing Stress

  exhibit this relative movement of longitudinal elements after the slabs are glued indicates the presence of *shearing stresses* on longitudinal planes.

  – Evaluation of these shearing stresses will be determined in the next couple of viewgraphs.

Shearing Stress in Beams

- Development of Shear Stress Formula

  Consider the free-body diagram of the short portion of the beam of Figs. 23 and 24a with a rectangular cross section shown in Fig 24b.

  From this figure,

$$dA = t\,dy \qquad (43)$$

$$dF = \sigma\,dA \qquad (44)$$

Shearing Stress in Beams

- Development of Shear Stress Formula

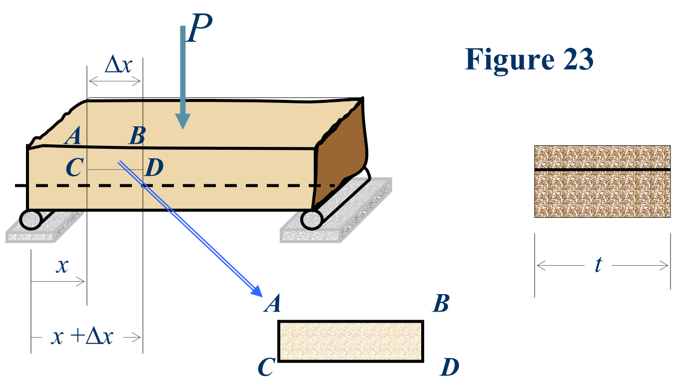


Figure 23

Shearing Stress in Beams

- Development of Shear Stress Formula

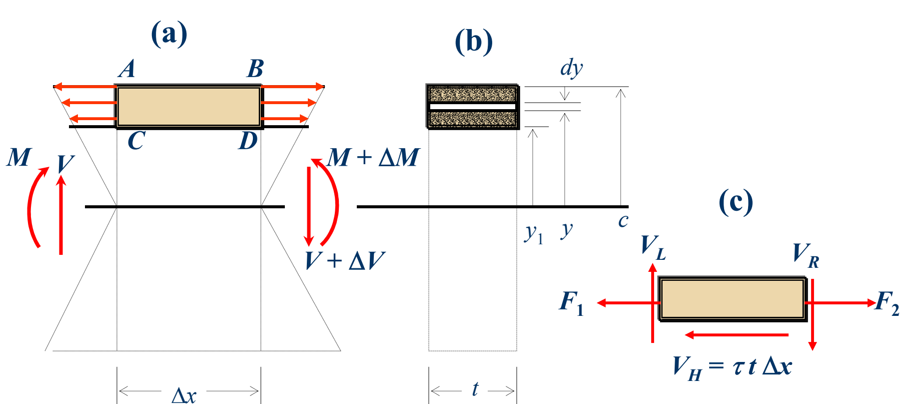


Figure 24

Shearing Stress in Beams

- Development of Shear Stress Formula

The resultant of these differential forces is $F = \int \sigma dA$ integrated over the area of the cross section, where σ is the flexural stress at a distance *y* from the neutral axis (surface) and is given by

$$\sigma = -\frac{My}{I} \qquad (45)$$

Shearing Stress in Beams

- Development of Shear Stress Formula

Therefore, the resultant normal force F_1 on the left end of the segment from y_1 to the top of the beam is

$$F_1 = -\frac{M}{I}\int y\, dA = -\frac{M}{I}\int_{y_1}^{c} y(t\, dy) \qquad (46)$$

Similarly, the resultant F_2 on the right side of the element is

$$F_2 = -\frac{M+\Delta M}{I}\int y\, dA = -\frac{M+\Delta M}{I}\int_{y_1}^{c} y(t\, dy) \qquad (47)$$

Shearing Stress in Beams

- Development of Shear Stress Formula

In reference to Fig. 24c, a summation of forces in the horizontal direction yields

$$\begin{aligned} V_H &= F_2 - F_1 \\ &= -\frac{M + \Delta M}{I}\int_{y_1}^{c} y(t\,dy) - \frac{M}{I}\int_{y_1}^{c} y(t\,dy) \\ &= -\frac{\Delta M}{I}\int_{y_1}^{c} y(t\,dy) \end{aligned} \qquad (47)$$

Shearing Stress in Beams

- Development of Shear Stress Formula

The average shearing stress τ_{avg} is the horizontal force V_H divided by the horizontal shear area $A_s = t\,\Delta x$ between section *A* and *B*. Thus

$$\tau_{avg} = \frac{V_H}{A_s} = -\frac{\Delta M}{I(t\,\Delta x)}\int_{y_1}^{c} y(t\,dy) \qquad (48)$$

Shearing Stress in Beams

- Development of Shear Stress Formula

  In the limit as Δx approaches zero, we have

$$\tau = \lim_{\Delta x \to 0} -\frac{\Delta M}{I(t\,\Delta x)} \int_{y_1}^{c} y(t\,dy)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x}\left(-\frac{1}{It}\right) \int_{y_1}^{c} y(t\,dy) \qquad (49)$$

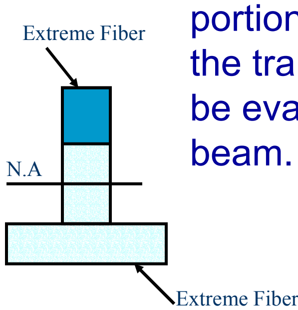
$$= \frac{dM}{dx}\left(-\frac{1}{It}\right) \int_{y_1}^{c} ty\,dy$$

Shearing Stress in Beams

- Development of Shear Stress Formula
  - Recall that equation 42 relates the bending moment with the shear force as $V = dM/dx$. In other words, the shear force V at the beam section where the stress is to be evaluated is given by Eq. 42.
  - The integral $\int_{y_1}^{c} ty\,dy$ of Eq. 49 is called the first moment of the area.

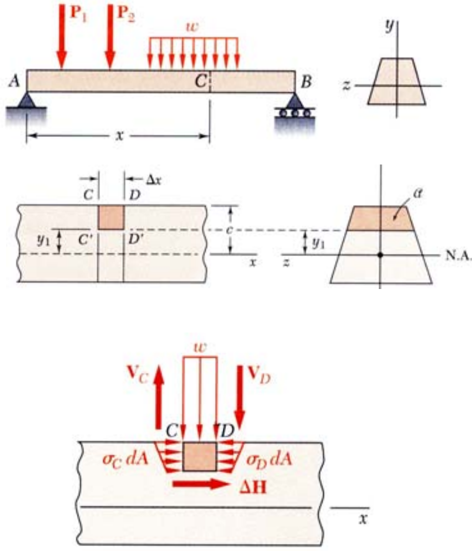
Shearing Stress in Beams

- Development of Shear Stress Formula
  - The integral $\int_{y_1}^{c} ty\,dy$ is usually given the symbol *Q.* Therefore, *Q* is the first moment of the portion of the cross-sectional area between the transverse line where the stress is to be evaluated and the extreme fiber of the beam.



$$Q = \int_{y_1}^{c} ty\,dy \qquad (50)$$

Shear on the Horizontal Face of a Beam Element



- Consider prismatic beam
- For equilibrium of beam element

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_D)\,dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y\,dA$$

- Note,

$$Q = \int_A y\,dA$$

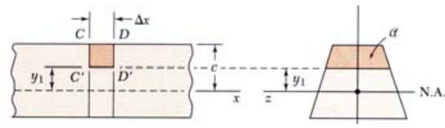
$$M_D - M_C = \frac{dM}{dx}\Delta x = V\,\Delta x$$

- Substituting,

$$\Delta H = \frac{VQ}{I}\Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \underline{shear\ \ flow}$$

Shear on the Horizontal Face of a Beam Element



- Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = shear\ \ flow$$

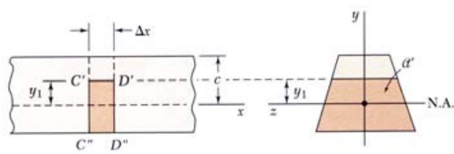
- where

$$Q = \int_A y\,dA$$

$$= \text{first moment of area above } y_1$$

$$I = \int_{A+A'} y^2 dA$$

$$= \text{second moment of full cross section}$$



- Same result found for lower area

$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = -q'$$

$$Q + Q' = 0$$

$$= \text{first moment with respect to neutral axis}$$

$$\Delta H' = -\Delta H$$

Shearing Stress in Beams

- Example 12

  Determine the first moment of area *Q* for the areas indicated by the shaded areas *a* and *b* of Fig. 25.

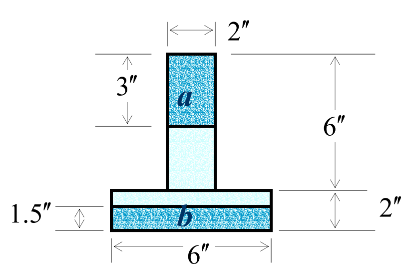


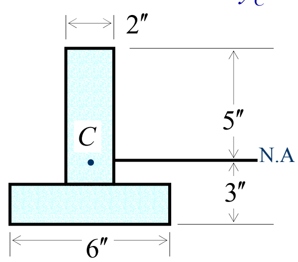
Figure 25

Shearing Stress in Beams

- Example 12 (cont'd)

First, we need to locate the neutral axis from the bottom edge:

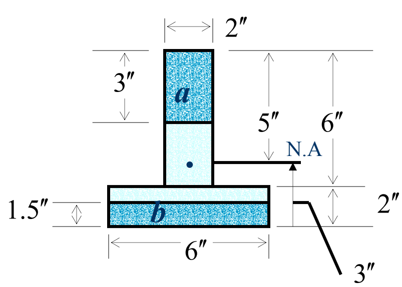
$$y_C = \frac{(1)(2\times 6)+(2+3)(2\times 6)}{2\times 6+2\times 6} = \frac{72}{24} = 3'' \text{ from base}$$



Shearing Stress in Beams

- Example 12 (cont'd)

The first moments of area Q_a and Q_b are found as follows:

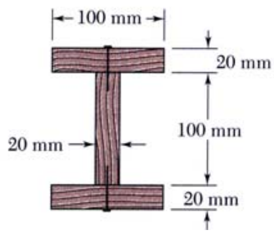


$$Q_a = (5-1.5)[3\times 2] = 21 \text{ in}^3$$

$$Q_b = \left(3-\frac{1.5}{2}\right)[1.5\times 6] = 20.25 \text{ in}^3$$

Shearing Stress in Beams

- Example 13

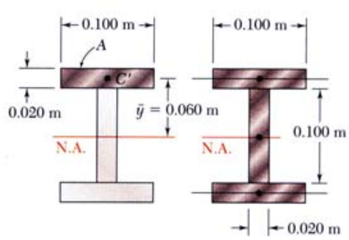


A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is V = 500 N, determine the shear force in each nail.

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.
- Calculate the corresponding shear force in each nail.

Example 13 (cont'd)



$$Q = A\bar{y} = (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m}) = 120 \times 10^{-6}\,\text{m}^3$$

$$I = \tfrac{1}{12}(0.020\,\text{m})(0.100\,\text{m})^3 + 2[\tfrac{1}{12}(0.100\,\text{m})(0.020\,\text{m})^3 + (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})^2] = 16.20 \times 10^{-6}\,\text{m}^4$$

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.

$$q = \frac{VQ}{I} = \frac{(500\text{N})(120 \times 10^{-6}\,\text{m}^3)}{16.20 \times 10^{-6}\,\text{m}^4} = 3704\,\text{N/m}$$

- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

$$F = (0.025\,\text{m})q = (0.025\,\text{m})(3704\,N/m$$

$$\boxed{F = 92.6\,\text{N}}$$

Shearing Stress in Beams

- Development of Shear Stress Formula

  Substituting for *dM/dx* of Eq. 42 and $Q = \int_{y_1}^{c} ty\,dy$ of Eq. 50 into Eq. gives

$$\tau = \frac{dM}{dx}\left(-\frac{1}{It}\right)\int_{y_1}^{c} ty\,dy$$

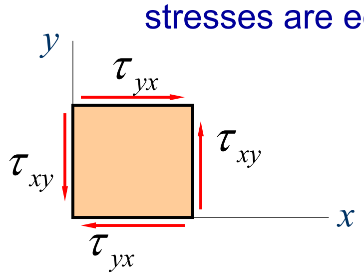
$$= (V)\left[-\frac{1}{It}\right]Q$$

$$\tau = -\frac{VQ}{It} \qquad (51)$$

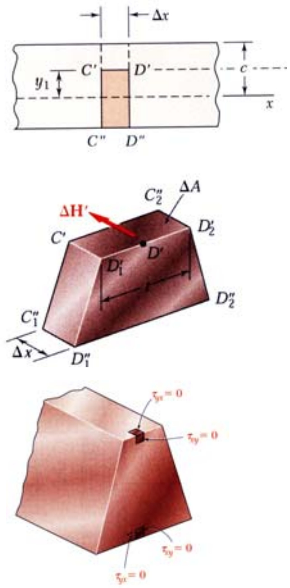
Shearing Stress in Beams

- Development of Shear Stress Formula

  Eq. 51 provides a formula to compute the horizontal (longitudinal) and vertical (transverse) shearing stresses at each point at a beam. These vertical and horizontal stresses are equivalent in magnitude.



Determination of the Shearing Stress in a Beam



- The *average* shearing stress on the horizontal face of the element is obtained by dividing the shearing force on the element by the area of the face.

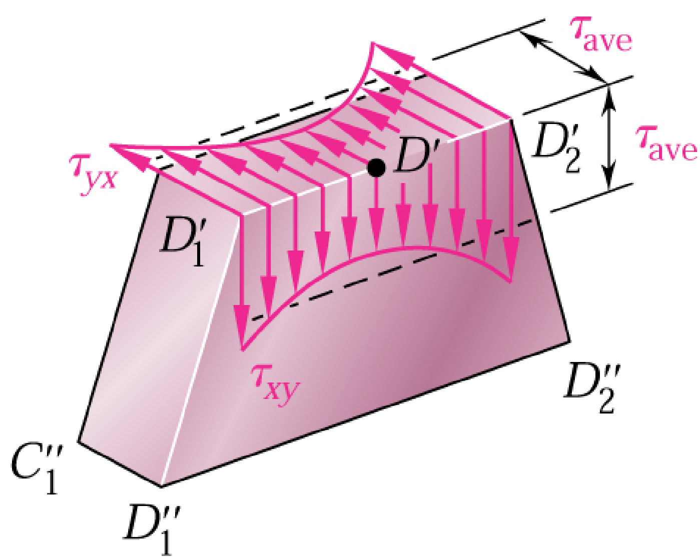
$$\tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{q\,\Delta x}{\Delta A} = \frac{VQ}{I}\frac{\Delta x}{t\,\Delta x}$$

$$= \frac{VQ}{It}$$

- On the upper and lower surfaces of the beam, $\tau_{yx} = 0$. It follows that $\tau_{xy} = 0$ on the upper and lower edges of the transverse sections.

- If the width of the beam is comparable or large relative to its depth, the shearing stresses at D_1 and D_2 are significantly higher than at D.

Shearing Stress in Beams



Shearing Stress in Beams

- Shearing Stress Formula

  At each point in the beam, the horizontal and vertical shearing stresses are given by

$$\tau = \frac{VQ}{It} \qquad (52)$$

Where
V = shear force at a particular section of the beam
Q = first moment of area of the portion of the cross-sectional area between the transverse line where the stress is to be computed.
I = moment of inertia of the cross section about neutral axis
t = average thickness at a particular location within the cross section

Shearing Stress in Beams

- Shearing Stress Formula

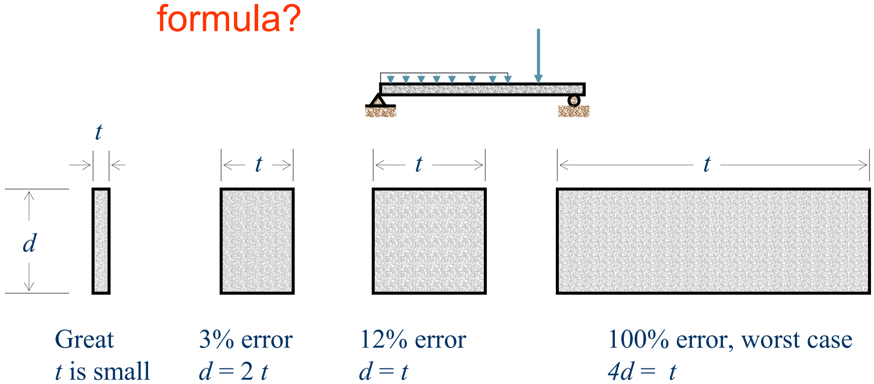
  How accurate is the shearing stress formula?

  - The formula is accurate if t is not too great.
  - For a rectangular section having a depth twice the width, the maximum stress as computed by more rigorous method is about 3% greater than that given by Eq. 52.
  - If the beam is square, the error is about 12%.
  - If the width is four times the depth, the error is about 100 %.

Shearing Stress in Beams

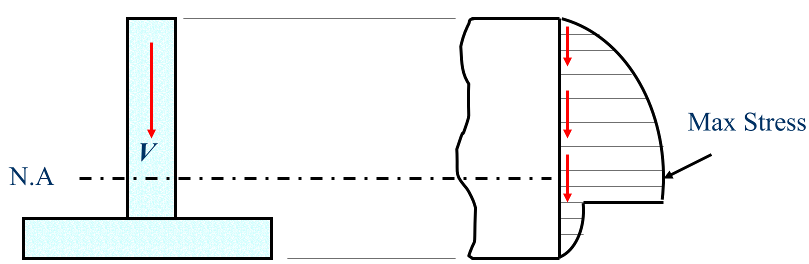
- Shearing Stress Formula

How accurate is the shearing stress formula?

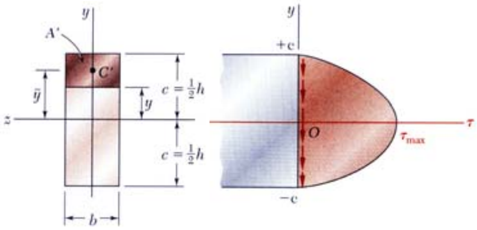


Shearing Stress in Beams

- Variation of Vertical Shearing Stress in the Cross Section



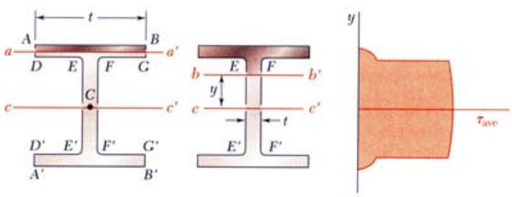
Shearing Stresses τ_{xy} in Common Types of Beams



- For a narrow rectangular beam,

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right)$$

$$\tau_{\max} = \frac{3}{2}\frac{V}{A}$$



- For American Standard (S-beam) and wide-flange (W-beam) beams

$$\tau_{ave} = \frac{VQ}{It}$$

$$\tau_{\max} = \frac{V}{A_{web}}$$

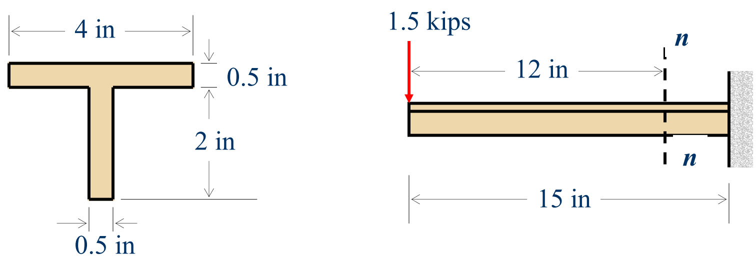
Shearing Stress in Beams

- Example 14

  A machine part has a T-shaped cross section and is acted upon in its plane of symmetry by the single force shown. Determine (a) the maximum compressive stress at section *n-n* and (b) the maximum shearing stress.

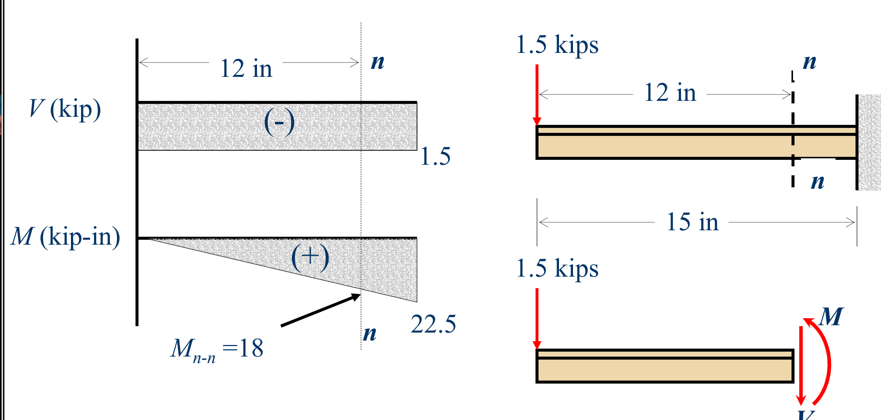
Shearing Stress in Beams

- Example 14 (cont'd)



Shearing Stress in Beams

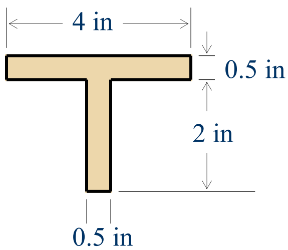
- Example 14 (cont'd)



Shearing Stress in Beams

- Example 14 (cont'd)

First, we need to locate the neutral axis. Let's make our reference from the bottom edge.



$$y_C = \frac{(1)(2\times0.5)+(2+0.25)(4\times0.5)}{2\times0.5+4\times0.5} = 1.833 \text{ in}$$

$$y_{\text{ten}} = 2.5 - 1.833 = 0.667 \text{ in} \qquad y_{\text{com}} = 1.833 \text{ in} = y_{\max}$$

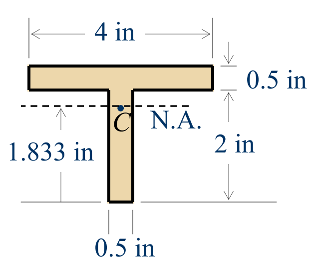
$$\text{Max. Stress} = \frac{M_r y_{\max}}{I_x}$$

Shearing Stress in Beams

- Example 14 (cont'd)

Next find the moment of inertia about the neutral axis:

$$I_x = \frac{0.5(1.833)^3}{3} + \frac{4(0.667)^3}{3} - \left[\frac{3.5(0.167)^3}{3}\right] = 1.417 \text{ in}^4$$



(a) Maximum normal stress is a compressive stress:

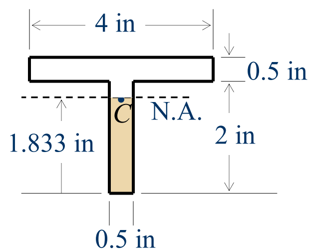
$$\sigma_{\max} = \frac{M_{n-n} c_{\max}}{I} = \frac{18(1.833)}{1.417} = 23.3 \text{ ksi (C)}$$

Shearing Stress in Beams

- Example 14 (cont'd)

(b) Maximum shearing stress:

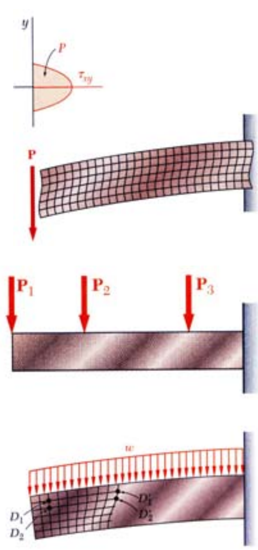
The maximum value of Q occurs at the neutral axis. Since in this cross section the width t is minimum at the neutral axis, the maximum shearing stress will occur there. Choosing the area below *a-a* at the neutral axis, we have



$$Q = \frac{1.833}{2}(0.5)(1.833) = 0.840\ \text{in}^3$$

$$\tau_{\max} = \frac{VQ}{It} = \frac{1.5(0.840)}{1.417(0.5)} = 1.778\ \text{ksi}$$

Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam



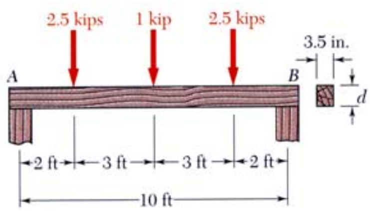
- Consider a narrow rectangular cantilever beam subjected to load P at its free end:

$$\tau_{xy} = \frac{3}{2}\frac{P}{A}\left(1 - \frac{y^2}{c^2}\right) \qquad \sigma_x = +\frac{Pxy}{I}$$

- Shearing stresses are independent of the distance from the point of application of the load.
- Normal strains and normal stresses are unaffected by the shearing stresses.
- From Saint-Venant's principle, effects of the load application mode are negligible except in immediate vicinity of load application points.
- Stress/strain deviations for distributed loads are negligible for typical beam sections of interest.

Shearing Stress in Beams

- Example 15



A timber beam is to support the three concentrated loads shown. Knowing that for the grade of timber used,

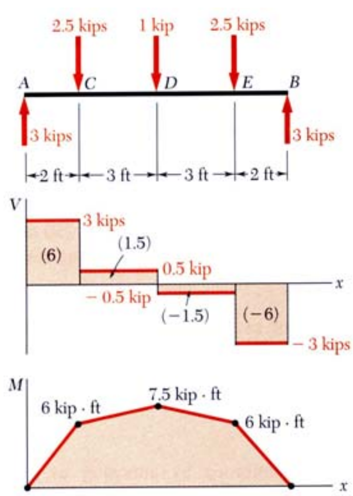
$$\sigma_{all} = 1800\,\text{psi} \qquad \tau_{all} = 120\,\text{psi}$$

determine the minimum required depth d of the beam.

SOLUTION:

- Develop shear and bending moment diagrams. Identify the maximums.
- Determine the beam depth based on allowable normal stress.
- Determine the beam depth based on allowable shear stress.
- Required beam depth is equal to the larger of the two depths found.

Shearing Stress in Beams



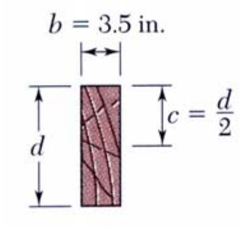
Example 15 (cont'd)

SOLUTION:

Develop shear and bending moment diagrams. Identify the maximums.

$$V_{max} = 3\,\text{kips}$$
$$M_{max} = 7.5\,\text{kip}\cdot\text{ft} = 90\,\text{kip}\cdot\text{in}$$

Example 15 (cont'd)



$$I = \tfrac{1}{12} b d^3$$

$$S = \frac{I}{c} = \tfrac{1}{6} b d^2$$

$$= \tfrac{1}{6}(3.5\,\text{in.})d^2$$

$$= (0.5833\,\text{in.})d^2$$

- Determine the beam depth based on allowable normal stress.

$$\sigma_{all} = \frac{M_{max}}{S}$$

$$1800\,\text{psi} = \frac{90 \times 10^3\,\text{lb} \cdot \text{in.}}{(0.5833\,\text{in.})d^2}$$

$$d = 9.26\,\text{in.}$$

- Determine the beam depth based on allowable shear stress.

$$\tau_{all} = \frac{3}{2}\frac{V_{max}}{A}$$

$$120\,\text{psi} = \frac{3}{2}\frac{3000\,\text{lb}}{(3.5\,\text{in.})d}$$

$$d = 10.71\,\text{in.}$$

- Required beam depth is equal to the larger of the two.

$$\boxed{d = 10.71\,\text{in.}}$$