LECTURE

7

Chapter
3.6

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SHAFTS: STATICALLY INDETERMINATE SHAFTS

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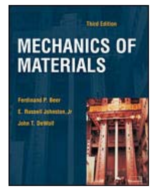

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ENES 220 – Mechanics of Materials

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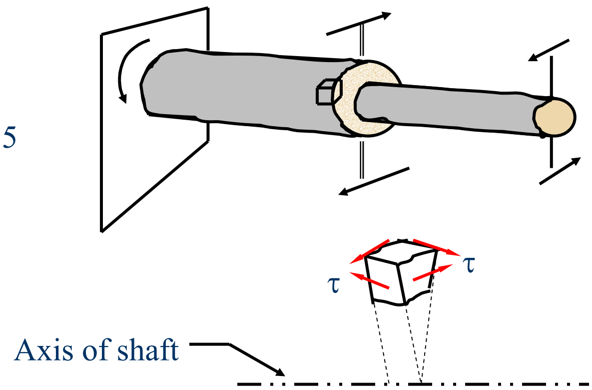
Stresses in Oblique Planes

- Up to this point, the stresses in a shaft has been limited to shearing stresses.
- This due to the fact that the selection of the element under study was oriented in such a way that its faces were either perpendicular or parallel to the axis of the shaft (see Fig. 15)

Stresses in Oblique Planes

Fig. 15



Stresses in Oblique Planes

- From our discussion of the torsional loading on a shaft, we know this loading produces shearing stresses τ in the faces perpendicular to the axis of the shaft.
- But due to equilibrium requirement, there are equal stresses on the faces formed by the two planes containing the axis of the shaft.

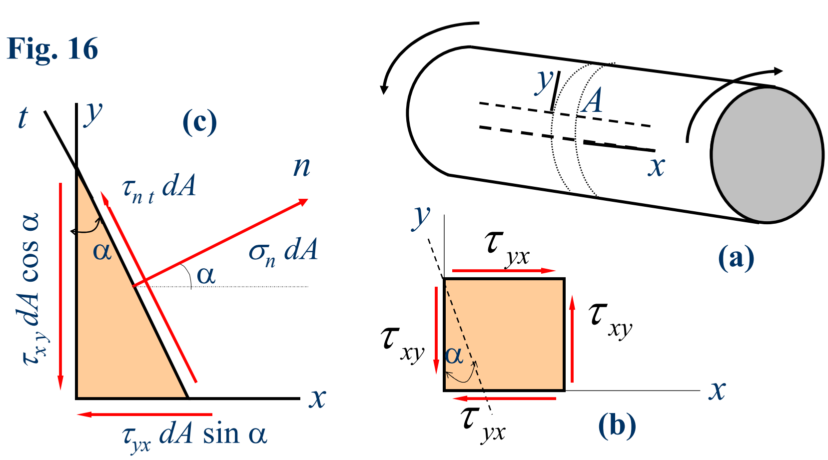
Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - It is necessary to make sure that whether the transverse plane is a plane of maximum shearing stress and whether there are other significant stresses induced by torsion.
  - Consider the following shaft (Fig. 16), which is subjected to a torque *T*.

Stresses in Oblique Planes

- Other Stresses Induced By Torsion

Fig. 16



Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - The stresses at point *A* in the shaft of Fig. 16a is analyzed.
  - A differential element taken from the shaft at point *A* and the stresses acting on transverse and longitudinal planes are shown in Fig. 16b.
  - The shearing stress τ_{xy} can be determined from

$$\tau_{xy} = \frac{Tc}{J}$$

Stresses in Oblique Planes

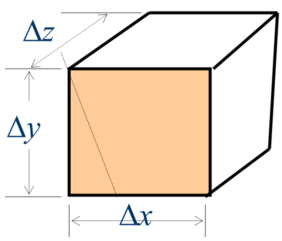
- Other Stresses Induced By Torsion
  - Let assume that differential element of Fig. 16b has length *dx*, height *dy*, and thickness *dz.*
  - If a shearing force $V_x = \tau_{xy}\, dx\, dy$ is applied to the top surface of the element, the equation of equilibrium $\sum F_x = 0$ then will require application of an opposite shear force V'_x at the bottom of the element.

Stresses in Oblique Planes

- Other Stresses Induced By Torsion

$$V_x = \tau_{yx} dx\, dz$$

$$V_y = \tau_{xy}\, dy\, dz$$



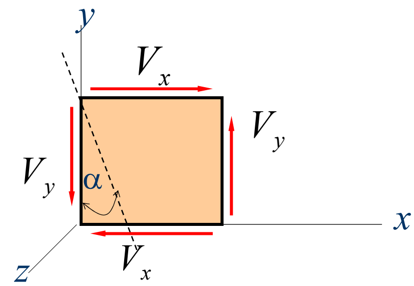


Fig. 17

Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - If $\sum F_x = 0$ then requires application of an opposite shear force V'_x at the bottom of the element, then it will the element subjected to a clockwise couple.
  - This clockwise couple must be balanced by counterclockwise couple composed of V_x applied to the vertical faces of the element.

Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - The application of the equilibrium moment equation $\sum M_z = 0$ gives

$$\tau_{yx}\left(dx\,dz\right)dy = \tau_{xy}\left(dy\,dz\right)dx$$

  - From which the important result

$$\boxed{\tau_{yx} = \tau_{xy}} \qquad (27)$$

Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - If the equations of equilibrium are applied to the free-body diagram of Fig. 16c (which is a wedge-shaped part of the differential element of Fig. 16b with *dA* being the area of the inclined face), the following results are obtained

$$(28)$$

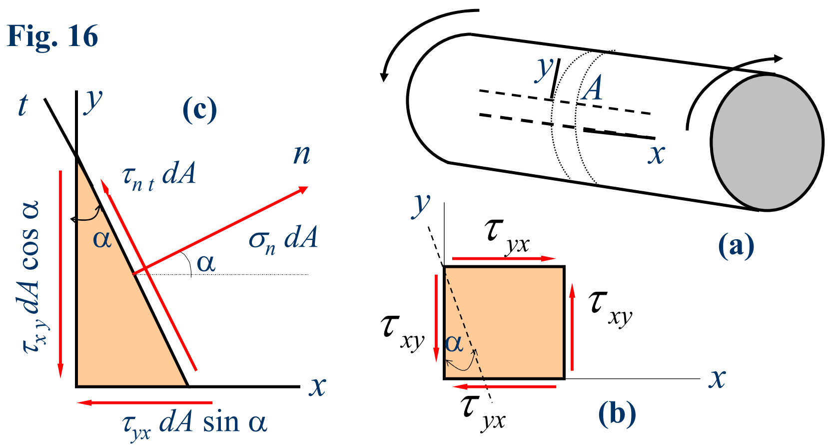
$$+\nwarrow \sum F_t = 0$$

$$\tau_{nt}dA - \tau_{xy}\left(dA\cos\alpha\right)\cos\alpha + \tau_{yx}\left(dA\sin\alpha\right)\sin\alpha = 0$$

Stresses in Oblique Planes

- Other Stresses Induced By Torsion

Fig. 16



Stresses in Oblique Planes

- Other Stresses Induced By Torsion

$$+\nwarrow \sum F_t = 0$$

$$\tau_{nt} dA - \tau_{xy}(dA\cos\alpha)\cos\alpha + \tau_{yx}(dA\sin\alpha)\sin\alpha = 0$$

From which

$$\tau_{nt} = \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) = \tau_{xy}\cos 2\alpha \qquad (29)$$

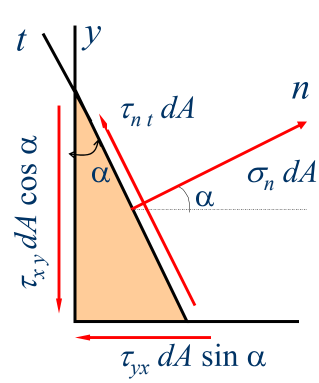


Fig. 16c

Stresses in Oblique Planes

- Other Stresses Induced By Torsion
  - Likewise, if we take summation of forces in the n direction (see Fig. 16c), then the results would be

$$+\nearrow\sum F_n = 0$$

$$\sigma_n dA - \tau_{xy}(dA\cos\alpha)\sin\alpha - \tau_{yx}(dA\sin\alpha)\cos\alpha = 0 \qquad (30)$$

Stresses in Oblique Planes

- Other Stresses Induced By Torsion

$$+\nearrow\sum F_t = 0$$

$$\sigma_n dA - \tau_{xy}(dA\cos\alpha)\sin\alpha - \tau_{yx}(dA\sin\alpha)\cos\alpha = 0$$

From which

$$\sigma_n = 2\tau_{xy}\sin\alpha\cos\alpha = \tau_{xy}\sin 2\alpha \qquad (31)$$

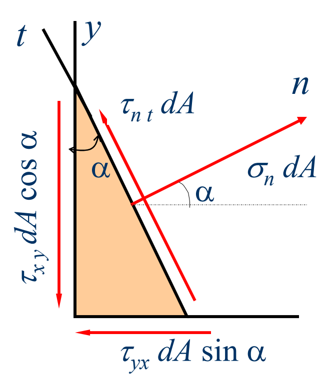


Fig. 16c

Stresses in Oblique Planes

- Maximum Normal Stress due to Torsion on Circular Shaft

  The maximum compressive normal stress $\sigma_{\max}$ can be computed from

$$\sigma_{\max} = \tau_{\max} = \frac{T_{\max} c}{J} \qquad (32)$$

Stresses in Oblique Planes

- Example 4

  A cylindrical tube is fabricated by butt-welding a 6 mm-thick steel plate along a spiral seam as shown. If the maximum compressive stress in the tube must be limited to 80 MPa, determine (a) the maximum torque *T* that can be applied and (b) the factor of safety with respect to the failure by fracture for the weld, when a torque of 12 kN.m is applied, if the ultimate strengths of the weld metal are 205 MPa in shear and 345 MPa in tension.

Stresses in Oblique Planes

- Example 4 (cont'd)

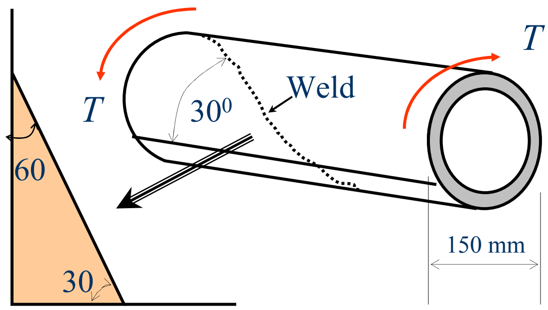


Fig. 18

Stresses in Oblique Planes

- Example 4 (cont'd)
  - (a) The polar moment of area for the cylindrical tube can be determined from Eq.14 as

$$J = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) = \frac{\tau}{2}\left[\left(\frac{150}{2}\right)^4 - \left(\frac{150-6}{2}\right)^4\right] = 14.096\times10^6\,\text{mm}^4$$

The maximum torque can be computed from Eq. 32 as

$$\sigma_{\max} = \frac{T_{\max} c}{J} \Rightarrow T_{\max} = \frac{\sigma_{\max} J}{c} = \frac{80\times10^6(14.096\times10^{-6})}{75\times10^{-3}}$$

$$= 15.036\times10^3\ \text{N}\cdot\text{m} = 15.036\ \text{kN}\cdot\text{m}$$

Stresses in Oblique Planes

- Example 4 (cont'd)

  (b)The normal stress σ_n and shear stress τ_{nt} on the weld surface are given by Eqs. 30 and 29 as

$$\sigma_n = \tau_{xy} \sin 2\alpha = \frac{Tc}{J} \sin 2\alpha = \frac{12 \times 10^3 (75 \times 10^{-3})}{14.096 \times 10^{-6}} \sin 2(60^0) = 55.29 \text{ MPa (T)}$$

$$\tau_{nt} = \tau_{xy} \cos 2\alpha = \frac{Tc}{J} \cos 2\alpha = \frac{12 \times 10^3 (75 \times 10^{-3})}{14.096 \times 10^{-6}} \cos 2(60^0) = -31.92 \text{ MPa}$$

Stresses in Oblique Planes

- Example 4 (cont'd)

  The factors of safety with respect to failure by fracture for the weld are

$$FS_\sigma = \frac{\sigma_{\text{ult}}}{\sigma_n} = \frac{345}{55.29} = 6.24$$

$$FS_\tau = \frac{\tau_{\text{ult}}}{\tau_{tn}} = \frac{205}{31.92} = 6.42$$

Statically Indeterminate Shafts

- Up to this point, all problems discussed are statically determinate, that is, only the equations of equilibrium were required to determine the torque T at any section of the shaft.
- It is often for torsionally loaded members to be statically indeterminate in real engineering applications.

Statically Indeterminate Shafts

- When this occurs, distortion equations involving angle of twist θ must written until the total number of equations agrees with the number of unknowns to be determined.
- A simplified angle of twist diagram will often be of great assistance in obtaining the correct equations.

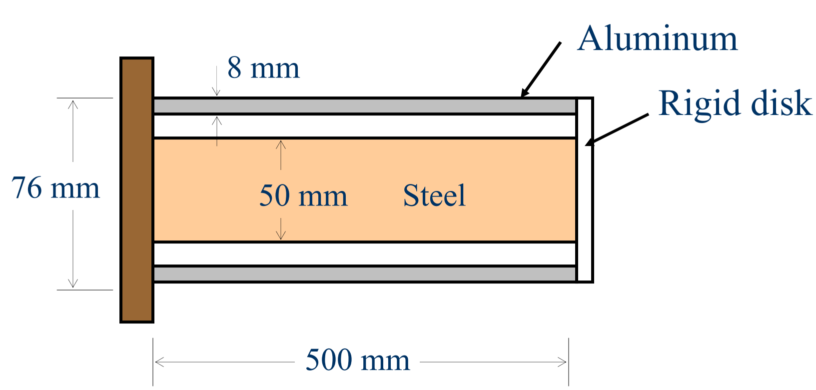
Statically Indeterminate Shafts

- Example 5

  A steel shaft and aluminum tube are connected to a fixed support and to a rigid disk as shown in the figure. Knowing that the initial stresses are zero, determine the minimum torque T_0 that may be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use G = 80 GPa for steel and G = 27 GPa for aluminum.

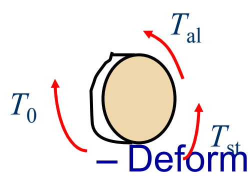
Statically Indeterminate Shafts

- Example 5 (cont'd)



Statically Indeterminate Shafts

- Example 5 (cont'd)
  - Free-body diagram for the rigid disk



From statics,

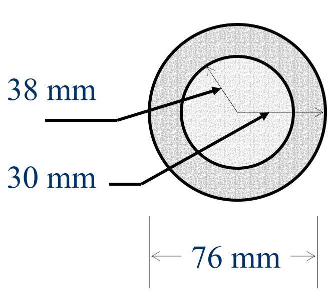
$$T_0 = T_{al} + T_{st} \qquad (39)$$

  - Deformation

$$\theta_{al} = \theta_{st} \Rightarrow \frac{T_{al}L_{al}}{J_{al}G_{al}} = \frac{T_{st}L_{st}}{J_{st}G_{st}} \qquad (40)$$

Statically Indeterminate Shafts

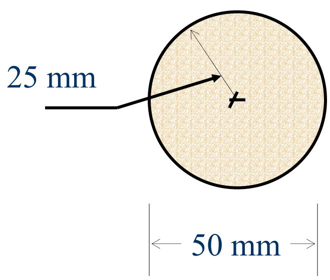
- Example 5 (cont'd)
  - Properties of the aluminum tube



$$G_{al} = 27\text{ GPa}$$
$$r_i = 30\text{ mm} = 0.030\text{ m}$$
$$r_o = 38\text{ mm} = 0.038\text{ m}$$
$$J_{al} = \frac{\pi}{2}\left[(0.038)^4 - (0.030)^4\right] =$$
$$= 2.003 \times 10^{-6}\text{ m}^4$$

Statically Indeterminate Shafts

- Example 5 (cont'd)
  - Properties of the steel tube



$$G_{st} = 80\ \text{GPa}$$

$$c = 25\ \text{mm} = 0.025\ \text{m}$$

$$J_{st} = \frac{\pi}{2}\left[(0.025)^4\right] =$$

$$= 0.6136 \times 10^{-6}\ \text{m}^4$$

Statically Indeterminate Shafts

- Example 5 (cont'd)

  Substituting these input values in Eq. 40, gives

$$\frac{T_{al} L_{al}}{J_{al} G_{al}} = \frac{T_{st} L_{st}}{J_{st} G_{st}}$$

$$\frac{T_{al}(0.5)}{2.003 \times 10^{-6}(27)} = \frac{T_{st}(0.5)}{0.6136 \times 10^{-6}(80)}$$

$$T_{st} = 0.908 T_{al} \qquad (41)$$

Statically Indeterminate Shafts

- Example 5 (cont'd)

  Let's assume that the requirement τ_{st} is less or to equal to 120 MPa, therefore

$$T_{st} = \frac{\tau_{st} J_{st}}{c_{st}} = \frac{120 \times 10^6 (0.6136 \times 10^{-6})}{0.025} = 2945 \text{ N} \cdot \text{m}$$

  From Eq. 39, we have

$$T_{st} = 0.908 T_{al}$$
$$2945 = 0.908 T_{al} \Rightarrow T_{al} = 3244 \text{ N} \cdot \text{m}$$

Statically Indeterminate Shafts

- Example 5 (cont'd)

  Let's check the maximum stress τ_{al} in aluminum tube corresponding to T_{al} = 3244 N·m:

$$\tau_{al} = \frac{T_{al} c_{al}}{j_{al}} = \frac{3244(0.038)}{2.003 \times 10^{-6}} = 61.5 \text{ MPa} < 70 \text{ MPa OK}$$

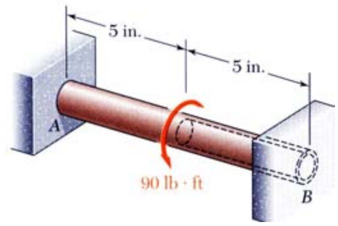
  Hence, the max permissible torque T_0 is computed from Eq. 39 as

$$T_0 = T_{al} + T_{st} = 3244 + 2945 = 6189 \text{ N} \cdot \text{m}$$
$$= 6.2 \text{ kN} \cdot \text{m}$$

Statically Indeterminate Shafts

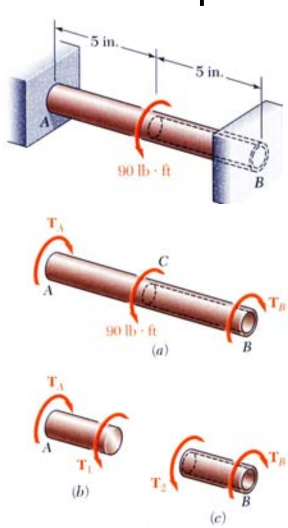
- Example 6

  A circular shaft *AB* consists of a 10-in-long, 7/8 in-diameter steel cylinder, in which a 5-in.-long, 5/8-in.-diameter cavity has been drilled from end *B*. The shaft is attached to fixed supports at both ends, and a 90 lb – ft torque is applied at its mid-section. Determine the torque exerted on the shaft by each of the supports.



Statically Indeterminate Shafts

- Example 6



- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

  $$T_A + T_B = 90\,\text{lb} \cdot \text{ft}$$

  which is not sufficient to find the end torques. The problem is statically indeterminate.
- Divide the shaft into two components which must have compatible deformations,

  $$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

  $$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90\,\text{lb} \cdot \text{ft}$$