LECTURE

27

Chapter 10.4

The McGraw-Hill Companies

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COLUMNS: BUCKLING (DIFFERENT ENDS)

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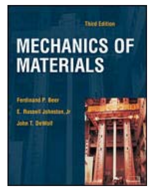

by

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ENES 220 – Mechanics of Materials

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Buckling of Long Straight Columns

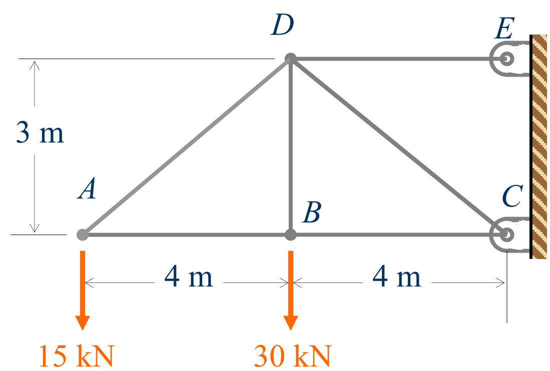
- Example 4

  A simple pin-connected truss is loaded and supported as shown in Fig. 12. All members of the truss are WT102 × 43 sections made of structural steel with a modulus of elasticity of 200 GPa and a yield strength of 250 MPa. Determine (a) the factor of safety with respect to failure by slip, and (b) the factor of safety with respect to failure by buckling.

Buckling of Long Straight Columns

- Example 4 (cont'd)

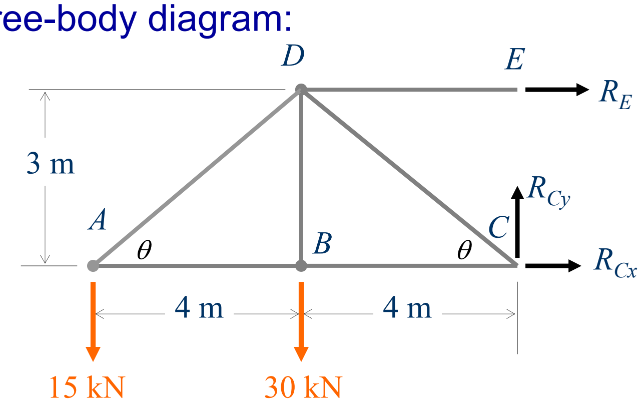
Figure 12



Buckling of Long Straight Columns

- Example 4 (cont'd)

  Free-body diagram:



Buckling of Long Straight Columns

- Example 4 (cont'd)

  Find the reactions R_E and R_C:

$$+\circlearrowleft \sum M_C = 15(8) + 30(4) - R_E(3) = 0; \Rightarrow R_E = 80 \text{ kN}$$

$$+\circlearrowleft \sum M_E = 15(8) + 30(4) + R_{Cx}(3) = 0; \Rightarrow R_{Cx} = -80 \text{ kN}$$

$$+\uparrow \sum F_y = 0;\ R_{Cy} - 30 - 15 = 0; \Rightarrow R_{Cy} = 45 \text{ kN}$$

Note that,

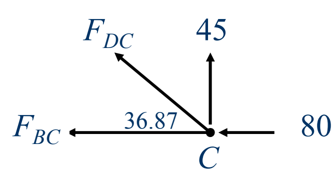
$$F_{DE} = R_E = \boxed{80 \text{ kN (T)}}$$

$$\theta = \tan^{-1}\frac{3}{4} = 36.87^0$$

Buckling of Long Straight Columns

- Example 4 (cont'd)

  At pin *C*:



$$+\uparrow \sum F_y = 0;\ F_{DC} \sin 36.87 + 45 = 0$$

$$F_{DC} = -75 \text{ kN} = \boxed{75 \text{ kN (C)}}$$

$$+\rightarrow \sum F_x = 0; -F_{BC} - 80 - F_{DC} \cos 36.87 = 0$$

$$\Rightarrow -F_{BC} - 80 - (-75)(0.8) = 0$$

$$\Rightarrow F_{BC} = -20 \text{ kN} = \boxed{20 \text{ kN (C)}}$$

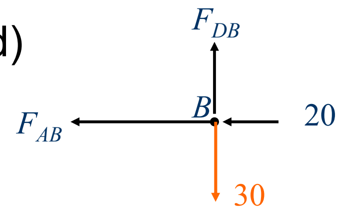
Buckling of Long Straight Columns

- Example 4 (cont'd)

At joint *B*:

$F_{DB} = 30\text{ kN}$

$F_{AB} = -20\text{ kN} = 20\text{ kN (C)}$

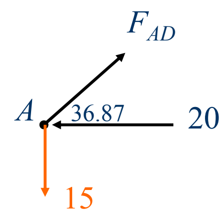


At Joint *A*:

$$+\uparrow \sum F_y = 0;$$

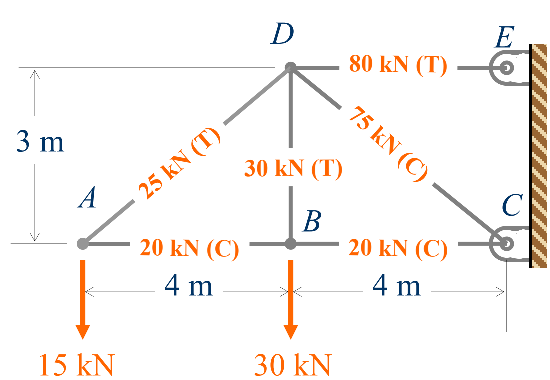
$$-15 + F_{AD}\sin 36.87 = 0$$

$$\therefore F_{AD} = 25\text{ kN (T)}$$



Buckling of Long Straight Columns

- Example 4 (cont'd)
  - Thus, the forces in the truss are as follows:



Buckling of Long Straight Columns

- Example 4 (cont'd)

(a) Factor of safety with respect to slip:

- For WT102 × 43, A = 5515 mm² (Appendix B)
- For member DE, the critical member that gives the largest force:

$$P_{\max} = \sigma_{\text{yield}} A = 250\times10^{6}\left(5515\times10^{-6}\right) = 1379\times10^{3}\ \text{N} = 1379\ \text{kN}$$

$$\therefore \text{FS} = \frac{P_{\max}}{F_{DE}} = \frac{1379}{80} = 17.24$$

Buckling of Long Straight Columns

- Example 4 (cont'd)

(b) Factor of safety with respect to failure by buckling:

- For WT102 × 43, $r_{\min}$ = 26.2 mm (Appendix B)
- For member DC, the critical member that gives the largest compressive force:

$$\frac{L}{r_{\min}} = \frac{5{,}000}{26.2} = 190.84\ \text{(slender)}$$

$$P_{cr} = \frac{\pi^2 EA}{(L/r_{\min})^2} = \frac{\pi^2\left(200\times10^{9}\right)\left(5515\times10^{-6}\right)}{(190.84)^2} = 298.9\ \text{kN}$$

$$\therefore \text{FS} = \frac{P_{cr}}{F_{DC}} = \frac{298.9}{75} = 3.99 \cong 4$$

Effects of Different Idealized End Conditions

- General Notes On Column Buckling
  1. Boundary conditions other than simply-supported will result in different critical loads and mode shapes.
  2. The buckling mode shape is valid only for small deflections, where the material is still within its elastic limit.
  3. The critical load will cause buckling for slender, long columns. In contrast, failure will occur in short columns when the strength of material is exceeded. Between the long and short column

Effects of Different Idealized End Conditions

- General Notes On Column Buckling

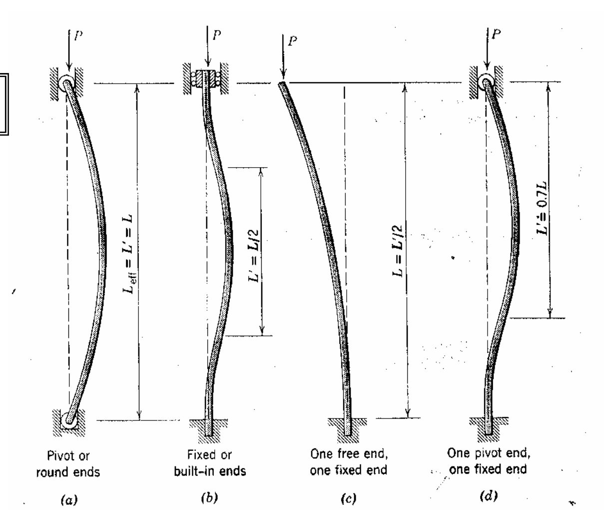
  limits, there is a region where buckling occurs after the stress exceeds the proportional limit but is still below the ultimate strength. These columns are classified as intermediate and their failure is called inelastic buckling.
  4. Whether a column is short, intermediate, or long depends on its geometry as well as the stiffness and strength of its material. This concept is addressed in the columns introduction page.

Effects of Different Idealized End Conditions

- The Concept of Effective Length
  - The Euler buckling formula, namely Eqs. 9 or 12 was derived for a column with pivoted ends.
  - The Euler equation changes for columns with different end conditions, such as the four common ones found in Figs.13 and 14.
  - While it is possible to set up the differential equation with appropriate boundary

Effects of Different Idealized End Conditions

Figure 13



Effects of Different Idealized End Conditions

Figure 14

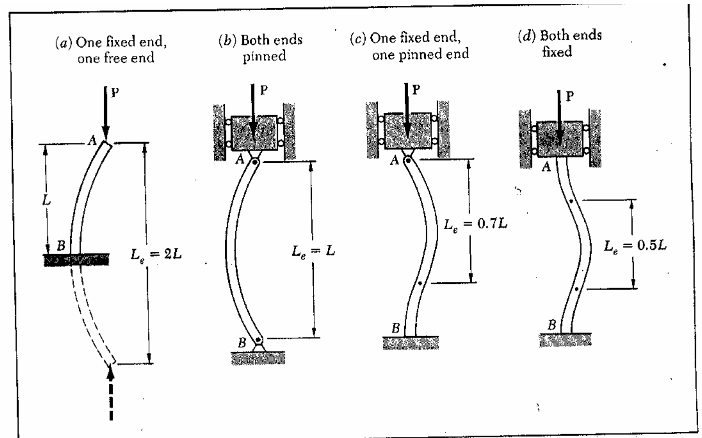


Fig. 11.17 Effective length of column for various end conditions. (Beer and Johnston 1992)

Effects of Different Idealized End Conditions

- The Effective Length Concept
  - Conditions to determine the Euler buckling formula for each case, a more common approach makes use of the concept of an "*effective length*".
  - The pivoted ended column, by definition, has zero bending moments at each end.
  - The length *L* in the Euler equation, therefore, is the distance between

Effects of Different Idealized End Conditions

- The Effective Length Concept
  - Successive points of zero bending moment.
  - All that is needed to modify the Euler column formula for use with other end conditions is to replace L by L'
  - L' is defined as the *effective length* of the column.

Effects of Different Idealized End Conditions

- The Effective Length Concept

  Definition:

  The effective length L' (or L_e) of a column is defined as the distance between successive inflection points or points of zero moment.

Effects of Different Idealized End Conditions

- The Effective Length Concept

Based on the effective length concept, the Euler Buckling load formula becomes

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad (16a)$$

or

$$P_{cr} = \frac{\pi^2 EA}{\left(L_e / r\right)^2} \qquad (16b)$$

$L_e = L' =$ effective length

Effects of Different Idealized End Conditions

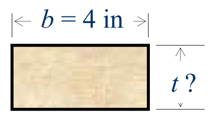
- Example 5

What is the least thickness a rectangular wood plank 4 in. wide can have, if it is used for a 20-ft column with one end fixed and one end pivoted, and must support an axial load of 1000 lb? Use a factor of safety (FS) of 5. The modulus of elasticity of wood is 1.5×10^6 psi.

Effects of Different Idealized End Conditions

- Example 5 (cont'd)

The rectangular cross section is



$$I = \frac{bt^3}{12}$$

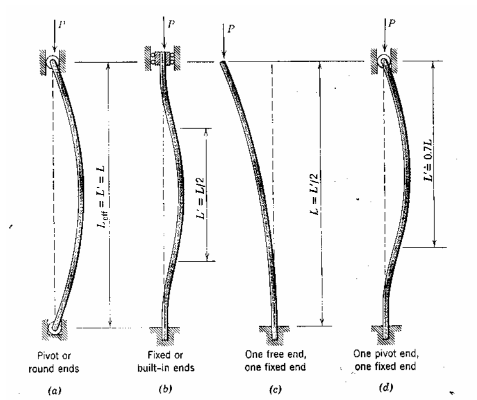
For the end conditions specified in the problem, Fig. 13d (or Fig. 14c) gives

$$L_e = 0.7L$$

Effects of Different Idealized End Conditions

- Example 5 (cont'd)

Figure 13



Effects of Different Idealized End Conditions

- Example 5 (cont'd)

Since *FS* = 5 is required,

$$P_{cr} = (FS)P = 5(1000) = 5{,}000 \text{ lb}$$

but

$$P_{cr} = \frac{\pi^2 EI}{L_e} = \frac{\pi^2(1.5\times10^6)\left(\frac{4t^3}{12}\right)}{[0.7(20\times12)]^2} = 5{,}000$$

From which

$$\boxed{t = 3.06 \text{ in}}$$

Effects of Different Idealized End Conditions

- Example 6

An L102 × 76 × 6.4-mm aluminum alloy (*E* = 70 GPa) angle is used for fixed-end, pivoted-end column having an actual length of 2.5 m. Determine the maximum safe load for the column if a factor of safety 1.75 with respect to failure by buckling is specified.

Effects of Different Idealized End Conditions

- Example 6 (cont'd)

  For fixed-end, pivoted-end column, Fig. 13d (or Fig. 14c) gives the equivalent length as

$$L_e = 0.7L = 0.7(2.5) = 1.75\ \text{m} = 1{,}750\ \text{mm}$$

  For L102 × 76 × 6.4 section (see Fig. 15 or Appendix B of textbook):

$$A = 1090\ \text{mm}^2$$

$$r_{\min} = 16.5\ \text{mm}$$

Figure 15

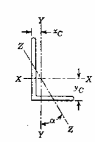


TABLE B-10 Unequal Leg Angles (SI Units)

Size and Thickness	Mass (kg/m)	Area (mm²)	AXIS X–X: I (10^6 mm⁴)	AXIS X–X: S (10^3 mm³)	AXIS X–X: r (mm)	AXIS X–X: y_C (mm)	AXIS Y–Y: I (10^6 mm⁴)	AXIS Y–Y: S (10^3 mm³)	AXIS Y–Y: r (mm)	AXIS Y–Y: x_C (mm)	AXIS Z–Z: r (mm)	AXIS Z–Z: Tan α
L229 × 102 × 15.9	39.1	4985	27.0	188	73.7	85.3	3.46	43.4	26.4	21.8	21.5	0.216
× 12.7	31.7	4030	22.1	153	74.2	84.1	2.88	35.6	26.7	20.6	21.7	0.220
L203 × 152 × 25.4	65.8	8385	33.6	247	63.2	67.3	16.1	146	43.9	41.9	32.5	0.543
× 19.1	50.3	6415	26.4	192	64.3	65.0	12.8	113	44.7	39.6	32.8	0.551
× 12.7	34.2	4355	18.4	131	65.0	62.7	9.03	78.5	45.5	37.3	33.0	0.558
L203 × 102 × 25.4	55.7	7095	29.0	231	64.0	77.5	4.83	64.6	26.2	26.7	21.5	0.247
× 19.1	42.7	5445	22.9	179	64.8	74.9	3.90	50.3	26.7	24.2	21.6	0.258
× 12.7	29.2	3710	16.0	123	65.8	72.6	2.81	35.2	27.4	21.8	22.0	0.267
L178 × 102 × 19.1	39.0	4960	15.7	138	56.4	63.8	3.77	49.7	27.7	25.7	21.8	0.324
× 12.7	26.6	3385	11.1	95.2	57.2	61.5	2.72	34.7	28.2	23.3	22.1	0.335
× 9.5	20.2	2570	8.57	72.8	57.7	60.2	2.12	26.7	28.7	22.1	22.4	0.340
L152 × 102 × 19.1	35.1	4475	10.2	102	47.8	52.8	3.61	48.7	28.4	27.4	21.8	0.428
× 12.7	24.1	3065	7.24	71.0	48.5	50.5	2.61	34.1	29.2	25.1	22.1	0.440
× 9.5	18.3	3230	5.62	54.4	49.0	49.3	2.04	26.2	29.7	23.9	22.3	0.446
L152 × 89 × 12.7	22.8	2905	6.91	69.5	48.8	52.8	1.77	26.1	24.7	21.2	19.3	0.344
× 9.5	17.4	2205	5.37	53.1	49.3	51.8	1.39	20.2	25.1	20.0	19.5	0.350
L127 × 89 × 19.1	29.5	3750	5.79	70.1	39.4	44.5	2.31	36.4	24.8	25.3	19.0	0.464
× 12.7	20.2	2580	4.16	49.0	40.1	42.2	1.69	25.6	25.7	23.0	19.2	0.479
× 9.5	15.5	1970	3.24	37.5	40.6	40.9	1.32	19.8	25.9	21.9	19.4	0.486
× 6.4	10.4	1330	2.24	25.7	41.1	39.6	0.928	13.6	26.4	20.7	19.6	0.492
L127 × 76 × 12.7	19.0	2420	3.93	47.7	40.4	44.5	1.07	18.8	21.1	19.1	16.5	0.357
× 9.5	14.6	1845	3.07	36.7	40.9	43.2	0.849	14.6	21.5	17.9	16.6	0.364
× 6.4	9.82	1250	2.13	25.1	41.1	42.2	0.599	10.1	21.9	16.7	16.8	0.371
L102 × 89 × 12.7	17.7	2260	2.21	31.8	31.2	31.8	1.58	24.9	26.4	25.4	18.3	0.750
× 9.5	13.5	1725	1.74	24.4	31.8	30.7	1.23	19.2	26.9	24.3	18.5	0.755
× 6.4	9.22	1170	1.21	16.9	32.3	29.5	0.870	13.2	27.2	23.1	18.6	0.759
L102 × 76 × 12.7	16.5	2095	2.10	31.0	31.8	33.8	1.01	18.4	21.9	21.0	16.2	0.543
× 9.5	12.6	1600	1.65	23.9	32.0	32.5	0.799	14.2	22.3	19.9	16.4	0.551
× 6.4	8.63	1090	1.15	16.4	32.5	31.5	0.566	9.82	22.8	18.7	16.5	0.558
L89 × 76 × 12.7	15.2	1935	1.44	23.8	27.2	28.7	0.970	18.0	22.4	22.2	15.8	0.714

Effects of Different Idealized End Conditions

- Example 6 (cont'd)

Therefore,

$$\frac{L'}{r_{\min}} = \frac{1750}{16.5} = 106.06 \text{ (slender)}$$

$$P_{\max} = \frac{P_{cr}}{FS} = \frac{\dfrac{\pi^2 EA}{(L'/r_{\min})^2}}{FS} = \frac{\pi^2 EA}{FS(L'/r_{\min})^2}$$

$$= \frac{\pi^2 (70\times 10^9)(1090\times 10^{-6})}{1.75(106.06)^2} = 38{,}254.54 \text{ N} = \boxed{38.3 \text{ kN}}$$

Effects of Different Idealized End Conditions

- Example 7

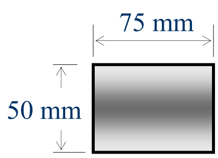
Determine the maximum load that a 50-mm × 75-mm × 2.5-m long aluminum alloy bar (E = 73 GPa) can support with a factor of safety of 3 with respect to failure by buckling if it is used as a fixed-end, pivoted column.

For fixed-end, pivoted column, Fig. 13d (or Fig. 14c) gives effective length, $L' = 0.7L = 0.7(2.5) = 1.75 \text{ m}$

Effects of Different Idealized End Conditions

- Example 7 (cont'd)

Computation of the section properties:



$$A = 75(50) = 3{,}750\ \text{mm}^2$$

$$I_{\min} = \frac{ba^3}{12} = \frac{75(50)^3}{12} = 0.7813\times10^6\ \text{mm}^4$$

$$r_{\min}\sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{0.7813\times10^6}{3750}} = 14.43\ \text{mm}$$

Effects of Different Idealized End Conditions

- Example 7 (cont'd)

Therefore, the slenderness ratio of the column can be obtained:

$$\frac{L'}{r_{\min}} = \frac{1.75\times10^3}{14.43} = 121.28\ \text{(slender)}$$

and

$$P_{\max} = \frac{P_{cr}}{FS} = \frac{\pi^2 EA}{FS(L'/r_{\min})} = \frac{\pi^2(73\times10^9)(3750\times10^{-6})}{3(121.28)^2} = \boxed{61.2\ \text{kN}}$$

Effects of Different Idealized End Conditions

- Example 8

A structural steel (E =29,000 ksi) column 20 ft long must support an axial compressive load of 200 kip. The column can be considered pivoted at one end and fixed at the other end for bending about one axis and fixed at both ends for bending about the other axis. Select the lightest wide-flange or American standard section that can be used for the column.

Effects of Different Idealized End Conditions

- Example 8 (cont'd)

First case: fixed-end, pivoted column

$$L_{ex} = L'_x = 0.7(20) = 14 \text{ ft}$$

$$P_{cr} = \frac{\pi^2 E I_x}{L_x'^2}$$

$$I_x = \frac{P_{cr} L_x'^2}{\pi^2 E} = \frac{200(14 \times 12)^2}{\pi^2 (29{,}000)} = 19.72 \text{ in}^2$$

Effects of Different Idealized End Conditions

- Example 8 (cont'd)

Second case: fixed ends column

$$L_{ey} = L'_y = 0.5(20) = 10 \text{ ft}$$

$$I_y = \frac{P_{cr} L_y'^2}{\pi^2 E} = \frac{200(10 \times 12)^2}{\pi^2 (29{,}000)} = 10.06 \text{ in}^2$$

Use a W254 × 33 section

Effects of Different Idealized End Conditions

- Example 9

A 25 mm-diameter tie rod *AB* and a pipe strut *AC* with an inside diameter of 100 mm and a wall thickness of 25 mm are used to support a 100-kN load as shown in Fig. 16. Both the tie rod and the pipe strut are made of structural steel with modulus of elasticity of 200 GPa and a yield strength of 250 MPa. Determine

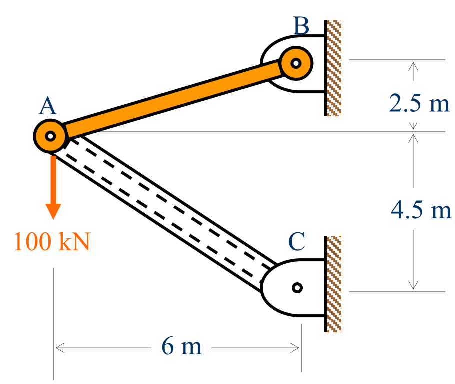
Effects of Different Idealized End Conditions

- Example 9 (cont'd)
  - (a) the factor of safety with respect to failure by slip.
  - (b) the factor of safety with respect to failure by buckling

Effects of Different Idealized End Conditions

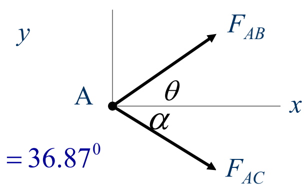
- Example 9 (cont'd)

Figure 16



Effects of Different Idealized End Conditions

- Example 9 (cont'd)



At joint A:

$$\theta = \tan^{-1}\frac{2.5}{6} = 22.62^0,\ \alpha = \tan^{-1}\frac{4.5}{6} = 36.87^0$$

$$+\rightarrow \sum F_x = 0;\ F_{AB}\cos\theta + F_{AC}\cos\alpha = 0$$

$$F_{AB}\cos 22.62 + F_{AC}\cos 36.87 = 0$$

$$F_{AB} = -0.8667 F_{AC} \qquad (17)$$

$$+\uparrow \sum F_x = 0;\ F_{AB}\sin\theta - F_{AC}\sin\alpha - 100 = 0$$

$$F_{AB}\sin 22.62 - F_{AC}\sin 36.87 - 100 = 0$$

$$F_{AB} = 1.56 F_{AC} + 260 \qquad (18)$$

Effects of Different Idealized End Conditions

- Example 9 (cont'd)

Substituting for F_{AB} in Eq. 17 into Eq. 18, gives

$$-0.8667 F_{AC} = 1.56 F_{AC} + 260$$

Thus,

$$F_{AC} = -107.14\ \text{kN} = 107.14\ \text{kN (C)}$$

From Eq. 18, with F_{AC} known, we have

$$F_{AB} = 1.56 F_{AC} + 260$$
$$= 1.56(-107.14) + 260 = 92.86\ \text{kN (T)}$$

Effects of Different Idealized End Conditions

- Example 9 (cont'd)

  Calculate the cross-sectional areas for *AB* and AC:

$$A_{AB} = \frac{\pi}{4}(25)^2 = 490.9\ \text{mm}^2, \quad A_{AC} = \frac{\pi}{4}\left[(150)^2 - (100)^2\right] = 9817\ \text{mm}^2$$

(a) FS due to Failure by Slip:

$$\sigma_{AB} = \frac{F_{AB}}{A_{AB}} = \frac{92.86\times10^3}{490.9\times10^{-6}} = 189.16\times10^6\ \text{N/m}^2 = 189.2\ \text{MPa}$$

$$\text{FS}_{\text{AB}} = \frac{250}{189.16} = \boxed{1.32}$$

Effects of Different Idealized End Conditions

- Example 9 (cont'd)

$$\sigma_{AC} = \frac{F_{AC}}{A_{AC}} = \frac{-107.14\times10^3}{9817\times10^{-6}} = -10.914\times10^6\ \text{N/m}^2 = 10.914\ \text{MPa (C)}$$

$$\text{FS}_{\text{AB}} = \frac{250}{10.914} = \boxed{22.9}$$

(b) FS due to Failure by Buckling:

- Member *AC* is the compression member with a compressive force of 107.14 kN:

$$I_{AC} = \frac{\pi}{64}\left[(150)^2 - (100)^2\right] = 19.942\times10^6\ \text{mm}^4$$

Effects of Different Idealized End Conditions

- Example 9 (cont'd)

  (b) FS for Buckling (cont'd):

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{19.942\times10^6}{9817}} = 45.07$$

$$\frac{L}{r} = \frac{\left(\sqrt{(6)^2 + (4.5)^2}\times 1000\right)}{45.07} = 166.41\,\text{(slender)}$$

$$P_{cr} = \frac{\pi^2 EA}{(L/r)^2} = \frac{\pi^2\left(200\times10^9\right)\left(9817\times10^{-6}\right)}{(166.41)^2} = 699.8\times10^3\ \text{N} \cong 700\ \text{kN}$$

$$\text{FS}_{\text{AC}} = \frac{P_{cr}}{F_{AC}} = \frac{700}{107.14} = \boxed{6.53}$$