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Third Edition

LECTURE

23

Chapter 7.5 – 7.6, 7.9

FAILURE CRITERIA: MULTIAXIAL STRESS STATES

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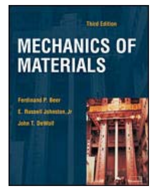

by

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SPRING 2003

ENES 220 – Mechanics of Materials

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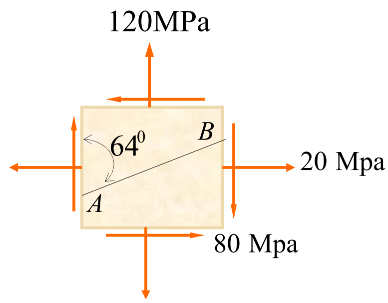
The Stress Transformation Equations for Plane Stress

- Example 7

  At a point in structural member subjected to plane stress there are normal and shearing stresses on horizontal and vertical planes through the point, as shown in Fig. 24. Use Mohr’s circle to determine (a) the principal stresses and maximum. shearing stress at the point, (b) the normal and shearing stresses on the inclined plane *AB* shown in the figure.

The Stress Transformation Equations for Plane Stress

- Example 7 (cont'd)



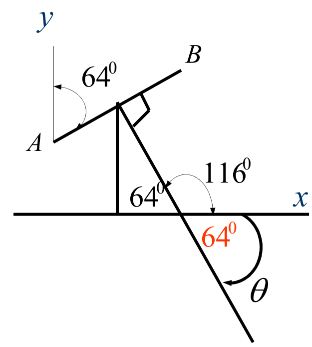


Fig.24

Mohr's Circle for Plane Stress

- Example 7 (cont'd)

The given values for use in drawing Mohr's circle are:

$\sigma_x = 20 \text{ MPa}$

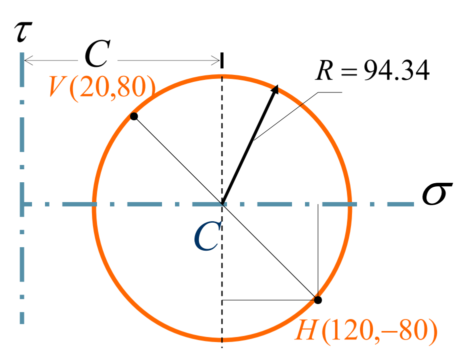
$\sigma_y = 120 \text{ MPa}$

$\tau_{xy} = \text{-}80 \text{ MPa}$

$\sigma_z = \sigma_{p3} = 0 \quad 2\theta = -2(64) = -128^0$

$C = \dfrac{20+120}{2} = 70 \text{ MPa}$

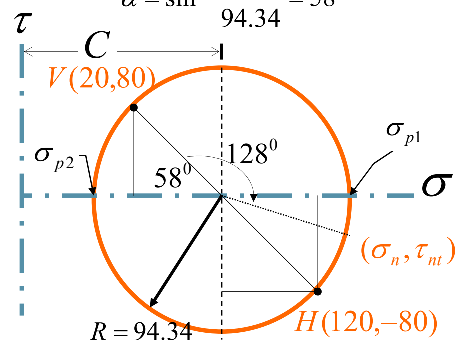
$R = \text{radius} = \sqrt{(120-70)^2 + (-80)^2} = 94.34$



Mohr's Circle for Plane Stress

- Example 7 (cont'd)

$$\alpha = \sin^{-1}\frac{80}{94.34} = 58^0$$



The principal stresses and maximum shearing stress can be computed as

(a)

$$\sigma_{p1} = C + R = 70 + 94.34 = 164.34 \text{ MPa}$$

$$\sigma_{p2} = C - R = 70 - 94.34 = -24.34 \text{ MPa}$$

$$\sigma_{p3} = 0$$

$$\tau_p = R = 94.34 \text{ MPa}$$

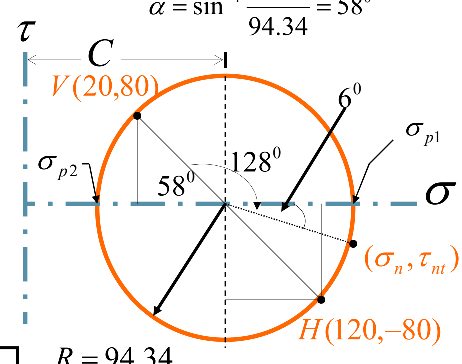
Since σ_{p1} and σ_{p1} have opposite sign,

$$\tau_{max} = \tau_p = 94.34 \text{ MPa}$$

Mohr's Circle for Plane Stress

- Example 7 (cont'd)

$$\alpha = \sin^{-1}\frac{80}{94.34} = 58^0$$



The normal and shearing stress on inclined plane can be computed as

(b)

$$\sigma_n = C + R\cos 6^0 = 70 + 94.34\cos 6^0 = \boxed{163.8 \text{ MPa}}$$

$$\tau_{nt} = R\sin 6^0 = 94.34\sin 6^0 = \boxed{9.86 \text{ MPa}}$$

Multiaxial Stress States

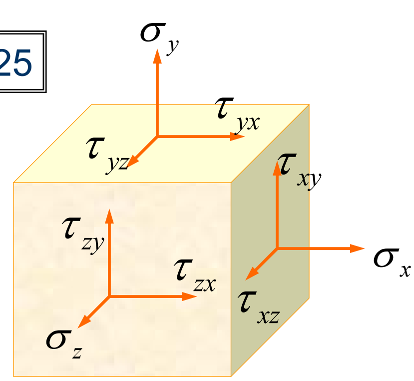
- Generalized Hooke's law
  - Hooke's law can be extended to include the biaxial and triaxial states of stress that often encounter in engineering applications.
  - Let's consider the differential element of the material subjected to biaxial state of normal stress (Figure 26)

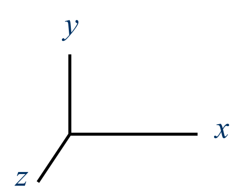
$$E = \frac{\sigma}{\varepsilon}$$

Multiaxial Stress States

- Generalized Hooke's Law

Fig.25



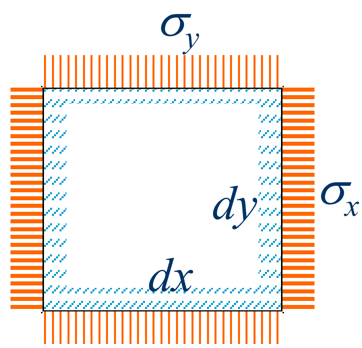


$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy} \qquad (29)$$
$$\tau_{zx} = \tau_{xz}$$

Multiaxial Stress States

- Generalized Hooke's Law

Fig.26



Multiaxial Stress States

- Generalized Hooke's Law
  - Shearing stresses have not been shown in the differential element of Fig. 26 because they do not produce changes in the lengths of sides of the element.
  - They only produce distortion of the element (angle changes), that can contributes to the angular strain.

Multiaxial Stress States

- Generalized Hooke's Law
  - The deformation of that element in the direction of the normal stresses, for a combined loading, can be determined by computing the deformations resulting from the individual stresses separately and adding the values obtained algebraically.
  - This procedure is based on the principle of superposition.

Multiaxial Stress States

- Generalized Hooke's Law
  - When applying the principle of superposition, the following conditions must be satisfied:
    - Each effect is linearly related to the load that produced it.
    - The effect of the first load does not scientifically change the effect of the second load.

Multiaxial Stress States

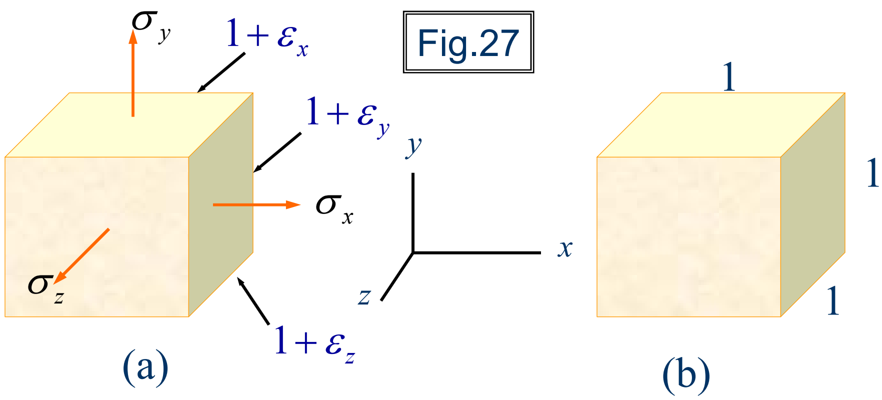
- Generalized Hooke's Law
  – The first condition is satisfied if the stresses do not exceed the proportional limit of the material.
  – The second condition is also satisfied if the deformations small so that the small changes in the areas of the faces of the element do not produce significant changes in the stresses.

Multiaxial Stress States

- Generalized Hooke's Law
  – Consider an element of material in the shape of cube as shown in Figure 27.
  – We may assume the side of the cube to be equal to unity, since it is always possible to select the side of cube as a unit length.
  – Under the given multiaxial loading, the element will deform into a rectangular parallelepiped of sides equal, respectively to

Multiaxial Stress States

- Generalized Hooke's Law



Multiaxial Stress States

- Generalized Hooke's Law

$$\begin{aligned} &1+\varepsilon_x \\ &1+\varepsilon_y \\ &1+\varepsilon_z \end{aligned} \qquad (30)$$

Where ε_x, ε_x, and ε_z denote the values of the normal strain in the directions of the three coordinate axes of Figure 27b.

Multiaxial Stress States

- Generalized Hooke's Law
  - Considering first the effect of the stress component σ_x, we know that this stress will cause a strain equal to σ_x/E in the *x*-direction, and strains equal to $(-\nu\,\sigma_x/E)$ in each of the *y* and *z* directions.
  - In a similar manner, if σ_y is applied separately, it will cause a strain equal to σ_y/E in the y- direction, and strains equal to $(-\nu\,\sigma_y/E)$ in the other directions.

Multiaxial Stress States

- Generalized Hooke's Law
  - Finally, the stress component σ_z will cause a strain equal to σ_z/E in the *z*- direction, and strains equal to $(-\nu\,\sigma_z/E)$ in each of the *x* and *y* directions.
  - Combining the results obtained, we conclude that the components of strain corresponding to the given multiaxial loading are

Multiaxial Stress States

- General State of Strain

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \qquad (31)$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

Multiaxial Stress States

- Generalized Hooke's Law
  - It can be shown based on the previous discussion and on Eq. 31 that if these equations are solved in terms of the strains, we could have two formulations or expressions for the components of normal stresses.
  - These two general formulations are provided in the next two slides.

Multiaxial Stress States

- State of Stress

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

(32)

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right)$$

These equations can be used to calculate normal stresses from measured or computed normal strains.

Multiaxial Stress States

- General State of Stress

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\left(\varepsilon_x + \varepsilon_z\right)\right] \qquad (33)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right]$$

Multiaxial Stress States

- Useful Relationship between G, E and ν

$$G = \frac{E}{2(1+\nu)} \tag{34}$$

Multiaxial Stress States

- Generalized Hooke's Law for Shearing Stress and Strain in Isotropic Materials

$$\begin{aligned}
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\
\tau_{yz} &= G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \\
\tau_{zx} &= G\gamma_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}
\end{aligned} \tag{35}$$

Multiaxial Stress States

- Example 8

At a point on the surface of a structural steel (E = 200 GPa and G = 76 GPa) machine part subjected to a biaxial state of stress, the measured strains were ε_x = +750 μm/m, ε_y = +350 μm/m, and γ_{xy} = -560 μrad. Determine the stresses σ_x, σ_y, and τ_{xy} at the point.

Multiaxial Stress States

- Example 8 (cont'd)

The given values are:

ε_x = +750 μm/m, ε_y = +350 μm/m, γ_{xy} = -560 μrad, E = 200 GPa, and G = 76 GPa

Using Eq. 34,

$$G = \frac{E}{2(1+\nu)} \Rightarrow \nu = \frac{1}{2}\left[\frac{E}{G} - 2\right]$$

$$\nu = \frac{1}{2}\left[\frac{200}{76} - 2\right] = 0.3158$$

Multiaxial Stress States

- Example 8 (cont'd)

  Using Eq. 32

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{200\times10^9}{1-(0.3158)^2}[750 + 0.3158(350)]\times10^{-6}$$
$$= 191.17\times10^6 \text{ N/m}^2 = \boxed{191.2 \text{ MPa (T)}}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = \frac{200\times10^9}{1-(0.3158)^2}[350 + 0.3158(750)]\times10^{-6}$$
$$= 130.37\times10^6 \text{ N/m}^2 = \boxed{130.4 \text{ MPa (T)}}$$

Multiaxial Stress States

- Example 8 (cont'd)

  The shearing stress τ_{xy} can be computed from the the following relationship:

$$\tau_{xy} = G\gamma_{xy}$$
$$= 76\times10^9(-560\times10^{-6})$$
$$= 42.56\times10^6 \text{ N/m}^2 = \boxed{42.6 \text{ MPa}}$$

Multiaxial Stress States

- Example 9

  Determine the state of strain that corresponds to the following state of stress at a point in a steel (E = 30,000 ksi and ν = 0.30) machine part: σ_x = 15,000 psi, σ_y = 5000 psi, σ_z = 7500 psi, τ_{xy} = 5500 psi, τ_{yz} = 4750 psi, and τ_{zx} = 3200 psi

Multiaxial Stress States

- Example 9 (cont'd)

  The given values are as follows:

$$\sigma_x = 15.0\text{ ksi}, \sigma_y = 5.0\text{ ksi}, \sigma_z = 7.5\text{ ksi}$$

$$\tau_{xy} = 5.5\text{ ksi}, \tau_{yz} = 4.75\text{ ksi}, \tau_{zx} = 3.2\text{ ksi}$$

$$E = 30{,}000\text{ ksi, and } \nu = 0.30$$

  Using Eqs. 31, the strains are

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = \frac{1}{30{,}000}\left[15 - 0.3(5 + 7.5)\right] = 375\ \mu\text{in/in}$$

Multiaxial Stress States

- Example 9 (cont'd)

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] = \frac{1}{30{,}000}\left[5 - 0.3(15 + 7.5)\right] = \boxed{-58.3\ \mu\text{in/in}}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{30{,}000}\left[7.5 - 0.3(15 + 5)\right] = \boxed{50\ \mu\text{in/in}}$$

Using Eq. 35

$$\tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \Rightarrow \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}$$

Multiaxial Stress States

- Example 9 (cont'd)

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = \frac{2(1+0.3))}{30{,}000}(5.5) = \boxed{477\ \mu\text{rad}}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz} = \frac{2(1+0.3))}{30{,}000}(4.75) = \boxed{412\ \mu\text{rad}}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx} = \frac{2(1+0.3))}{30{,}000}(3.2) = \boxed{277\ \mu\text{rad}}$$

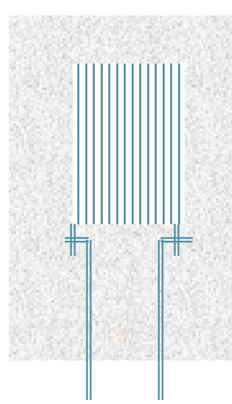
Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - Electrical resistance strain gages provide accurate measurements of normal strain.
  - The gage may consist of a length of 0.001 in-diameter wire arranged as shown in Figure 28 and cemented between two pieces of paper.

Multiaxial Stress States

- Strain Measurement and Rosette Analysis

Fig.28



Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - The wire or foil gage is centered to the material for which the strain is to be determine.
  - As the material is strained, the wires are lengthened or shortened.
  - This lengthening and shortening will cause changes in the electrical resistance.

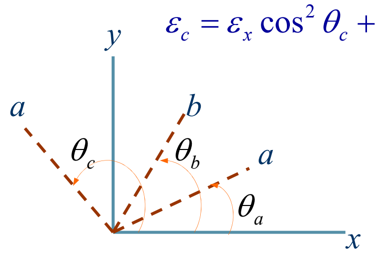
Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - The change in resistance can be measured and calibrated to provide normal strain
  - Shearing strains are often obtained by measuring normal strains in two or three different directions.
  - The shearing strains can be computed from normal strain data using the following equations:

Multiaxial Stress States

- Strain Measurement and Rosette Analysis

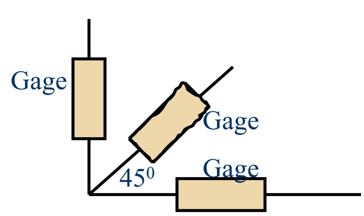
$$\varepsilon_a = \varepsilon_x \cos^2\theta_a + \varepsilon_{yx}\sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\vartheta_a$$
$$\varepsilon_b = \varepsilon_x \cos^2\theta_b + \varepsilon_{yx}\sin^2\theta_b + \gamma_{xy}\sin e\theta_b\cos\theta_b \quad (36)$$
$$\varepsilon_c = \varepsilon_x \cos^2\theta_c + \varepsilon_{yx}\sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$



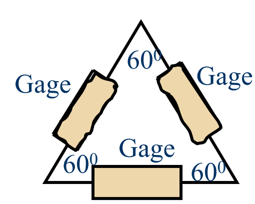
Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - Rosette Types

Fig.29



(a) 45^0 Rosette



(a) Delta Rosette

Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - In-plane strains and their Orientations

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{p1}, \varepsilon_{p2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (37)$$

$$\gamma_p = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  - The principal strain $\varepsilon_z = \varepsilon_{p3}$ can be determined from the measured data. From Eqs. 31 and 33,

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) = -\frac{\nu}{E}\left[\frac{E}{1-\nu^2}\right]\left[\left(\varepsilon_x + \nu\varepsilon_y\right) + \left(\varepsilon_y + \nu\varepsilon_x\right)\right]$$

$$= -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) \qquad (38)$$

Multiaxial Stress States

- Strain Measurement and Rosette Analysis
  – Out-of-plane principal Strain

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) \qquad (39)$$

Multiaxial Stress States

- Example 10

  At a point on the free surface of an aluminum alloy (E = 73 GPa and ν = 0.33) machine part, the strain rosette shown in Fig. 30 was used to obtain the following normal strain data: ε_a = +780μ, ε_b = +345μ, ε_c = -332μ. Determine (a) the strain components ε_x, ε_y, γ_{xy} and (2) the principal strains and maximum shearing strain.

Multiaxial Stress States

- Example 10 (cont'd)

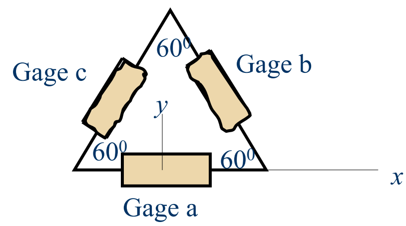


Figure.30. Delta Rosette

Multiaxial Stress States

- Example 10 (cont'd)

The given values are as follows:

$$\varepsilon_a = \varepsilon_x = +780\mu,\ \ \varepsilon_b = +345\mu\ (\theta = -60^0),\ \varepsilon_c = -332\mu\ (\theta = 60^0)$$

(a) $\varepsilon_n = \varepsilon_x \cos^2\theta_n + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta_n \cos\theta_n$

$$\varepsilon_b = 780\cos^2\left(-60^0\right) + \varepsilon_y \sin^2\left(-60^0\right) + \gamma_{xy}\sin\left(-60^0\right)_n \cos\left(-60^0\right) = +345\mu$$

$$\varepsilon_c = 780\cos^2\left(60^0\right) + \varepsilon_y \sin^2\left(60^0\right) + \gamma_{xy}\sin\left(60^0\right)_n \cos\left(60^0\right) = -332\mu$$

Solving above eqaution, yields

$$\boxed{\varepsilon_x = \varepsilon_a = +780\mu} \qquad \boxed{\varepsilon_y = -251\mu} \qquad \boxed{\gamma_{xy} = -782\mu\text{rad}}$$

Multiaxial Stress States

- Example 10 (cont'd)

$$\varepsilon_{p1}, \varepsilon_{p2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \frac{780 - 251}{2} \pm \sqrt{\left(\frac{780 + 251}{2}\right)^2 + \left(\frac{-782}{2}\right)^2}$$

Therefore,

$$\varepsilon_{p1} = 264.5 + 647 = \boxed{911.5\mu}$$

$$\varepsilon_{p2} = 264.5 - 647 = \boxed{-383\mu}$$

Multiaxial Stress States

- Example 10 (cont'd)

  Using Eq. 39, the third principal strain is

$$\varepsilon_{p3} = \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right) = \frac{0.33}{1-0.33}\left(780 - 251\right) = \boxed{-261\mu}$$

$$\gamma_p = \gamma_{\max} = 2(647) = \boxed{1294\ \mu\text{rad}}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{1}{2}\tan^{-1}\frac{-782}{780+251} = \boxed{18.59^0}$$

Multiaxial Stress States

- Example 11

  The strain rosette shown in Figure 31 was used to obtain normal strain data at a point on the free surface of a 2024-T4 aluminum alloy structural component. The gage readings were ε_a = +525μ, ε_b = +450μ, and ε_c = +1425μ. Determine

  (a) the strain components ε_x, ε_y, γ_{xy} at the point.

Multiaxial Stress States

- Example 11 (cont'd)

  (b) the stress components σ_x, σ_y, and γ_{xy}.

  (c) the principal stresses and maximum shearing stress at the point (in SI units).

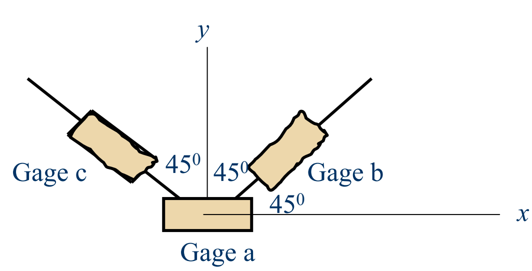


Figure.31

Multiaxial Stress States

- Example 11 (cont'd)

The given values are as follows:

$$\varepsilon_a = \varepsilon_x = +525\mu,\ \varepsilon_b = +450\mu\ (\theta = 45^0),\ \varepsilon_c = +1425\mu\ (\theta = -45^0)$$

$$\text{(a) } \varepsilon_n = \varepsilon_x \cos^2\theta_n + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta_n\cos\theta_n$$

$$\varepsilon_b = 525\cos^2(45^0) + \varepsilon_y\sin^2(45^0) + \gamma_{xy}\sin(45^0)_n\cos(45^0) = +450\mu$$

$$\varepsilon_c = 525\cos^2(-45^0) + \varepsilon_y\sin^2(-45^0) + \gamma_{xy}\sin(-45^0)_n\cos(-45^0) = +1425\mu$$

Solving above eqaution, yields

$$\boxed{\varepsilon_x = \varepsilon_a = +525\mu} \quad \boxed{\varepsilon_y = +1350\mu} \quad \boxed{\gamma_{xy} = -975\mu\text{rad}}$$

Multiaxial Stress States

- Example 11 (cont'd)

(b) For 2024 - T4 aluminum alloy : $E = 73$ GPa and $G = 28$ GPa

$$\text{Eq. 34 gives } \nu = \frac{1}{2}\left[\frac{E}{G} - 2\right] = \frac{1}{2}\left[\frac{73}{28} - 2\right] = 0.3036$$

Eqs 32 give

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y\right] = \frac{73\times10^9}{1-(0.3036)^2}\left[525 + 0.3036(1350)\right]\times10^{-6} = \boxed{75.2 \text{ MPa (T)}}$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x\right] = \frac{73\times10^9}{1-(0.3036)^2}\left[1350 + 0.3036(525)\right]\times10^{-6} = \boxed{121.4 \text{ MPa (T)}}$$

$$\tau_{xy} = G\gamma_{xy} = 28\times10^9\left(-975\times10^{-6}\right) = \boxed{-27.3 \text{ MPa}}$$

Multiaxial Stress States

- Example 11 (cont'd)

(c) Eq. 22a gives

$$\sigma_{p1,\,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{75.2 + 121.4}{2} \pm \sqrt{\left(\frac{75.2 - 121.4}{2}\right)^2 + (-27.3)^2}$$

$$\sigma_{p1} = 98.27 + 35.76 = \boxed{134 \text{ MPa}}$$

$$\sigma_{p2} = 98.27 - 35.76 = \boxed{62.5 \text{ MPa}}$$

$$\sigma_{p3} = \sigma_z = 0 \qquad \tau_p \boxed{= 35.8 \text{ MPa}} \text{(note} \neq \tau_{\max} \text{ because } \sigma_{p1} \text{ and } \sigma_{p2} \text{ have same sign)}$$

$$\tau_{\max} = \frac{1}{2}[\sigma_{\max} - \sigma_{\min}] = \frac{1}{2}[134 - 0] = \boxed{67 \text{ MPa}}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{1}{2}\tan^{-1}\frac{2(-27.3)}{75.2 - 121.4} \boxed{= 24.9^0}$$