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LECTURE

21

Chapter 7.1 – 7.3

# FAI LURE CRITERIA: STRESS TRANSFORMATION

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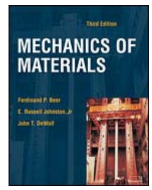


by
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**SPRING 2003**

**ENES 220 – Mechanics of Materials**

Department of Civil and Environmental Engineering
University of Maryland, College Park



## Combined Static Loading

- Introduction
  - Formulas were developed previously for determining normal and shearing stresses on a specific planes in
    - Axially loaded bars
    - Circular shafts, and
    - Beams
  - For example, the normal stress at a point on a cross section of a beam can be computed by the flexural formula.

# Combined Static Loading

- Introduction
  - The elastic flexural formula for normal stress is given by

$$\sigma_{max} = \frac{M_r c}{I} \qquad (1)$$

and

$$\sigma_x = \frac{M_r y}{I} \qquad (2)$$

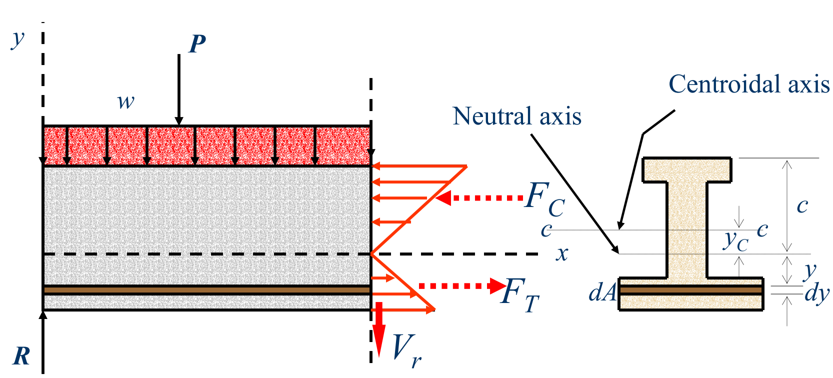


# Combined Static Loading

- Introduction

Figure 1

Distribution of Normal Stress in a Beam Cross Section



# Combined Static Loading

- Introduction
  - Also, as an example, the shearing stress at the same point on the cross section of the beam can be computed using the shearing stress formula:

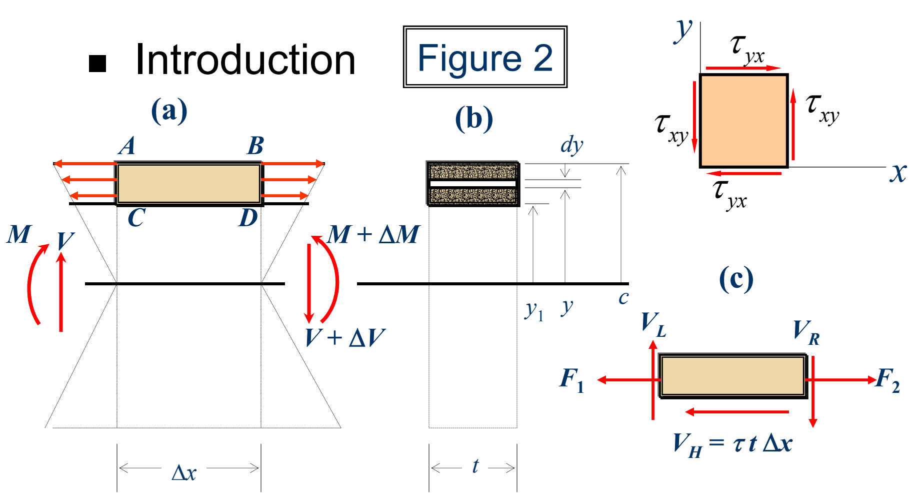
$$\tau = \frac{VQ}{It} \qquad (3)$$



# Combined Static Loading

- Introduction

Figure 2



# Combined Static Loading

- Introduction
  - Another example is the stress on circular shafts due to torsion.
  - The stress of a point on a circular shaft due to torsion can be computed from

$$\tau_{\max} = \frac{Tc}{J} \qquad (4)$$

$$\tau_{\rho} = \frac{T\rho}{J}$$



# Combined Static Loading

- Introduction
  - Distribution of Shearing Stress within the Circular Cross Section

$$\tau_c = \frac{Tc}{J}$$

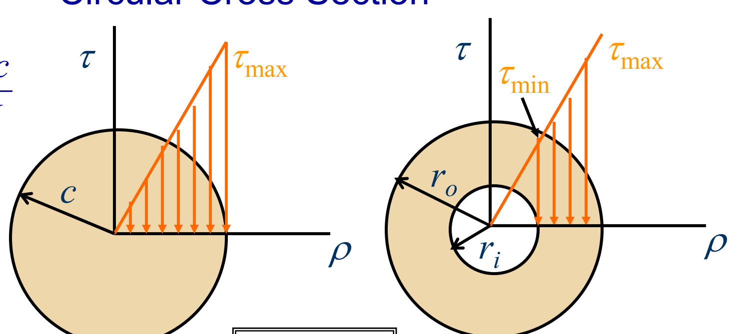
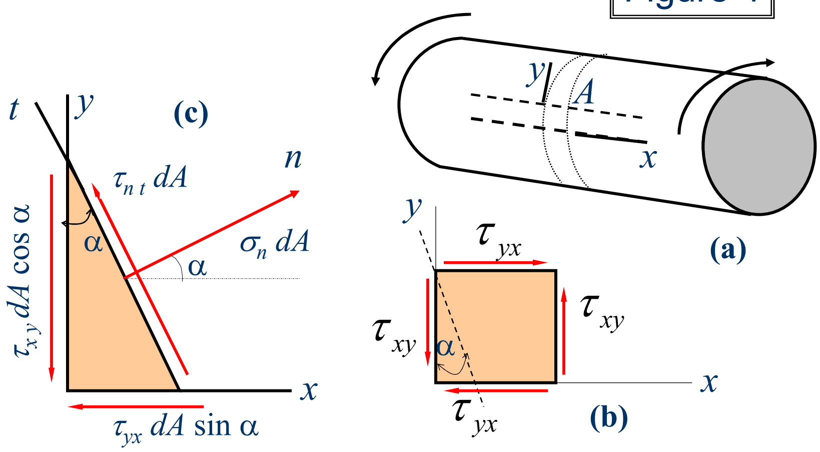



Figure 3

# Combined Static Loading

- Introduction

Figure 4






# Combined Static Loading

- Introduction

Figure 5

$$+\sum F_t = 0$$

$$\tau_{nt} dA - \tau_{xy}(dA\cos\alpha)\cos\alpha + \tau_{yx}(dA\sin\alpha)\sin\alpha = 0$$

From which

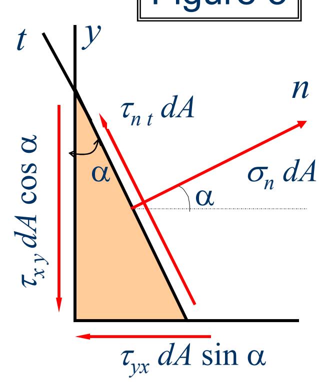
$$\tau_{nt} = \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) = \tau_{xy}\cos 2\alpha \qquad (5)$$

$$+\sum F_t = 0$$

$$\sigma_n dA - \tau_{xy}(dA\cos\alpha)\sin\alpha - \tau_{yx}(dA\sin\alpha)\cos\alpha = 0$$

From which

$$\sigma_n = 2\tau_{xy}\sin\alpha\cos\alpha = \tau_{xy}\sin 2\alpha \qquad (6)$$

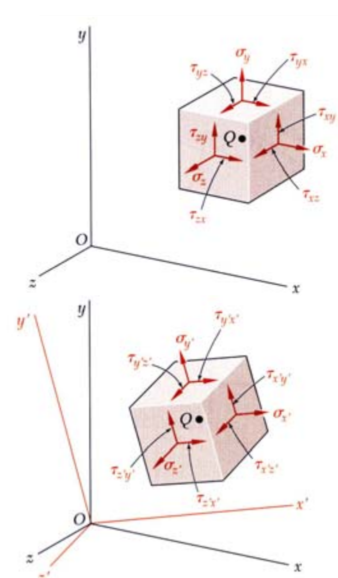


# Combined Static Loading

- Introduction
  - The method of finding the stress on a specified inclined plane as illustrated by free-body diagram of Fig. 5 is not suitable for determining the maximum normal and maximum shearing stresses.
  - A more general approach will be developed to handle this type of analysis.



# Introduction




- The most general state of stress at a point may be represented by 6 components,

  $\sigma_x, \sigma_y, \sigma_z$ normal stresses

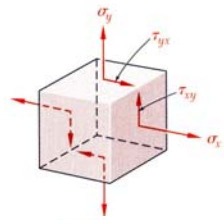
  $\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

  (Note: $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)

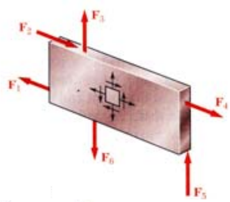
- Same state of stress is represented by a different set of components if axes are rotated.

- The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain.

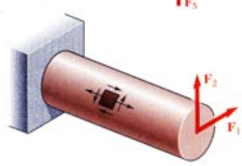
# Introduction




- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by $\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_z = \tau_{zx} = \tau_{zy} = 0.$




- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.




- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.





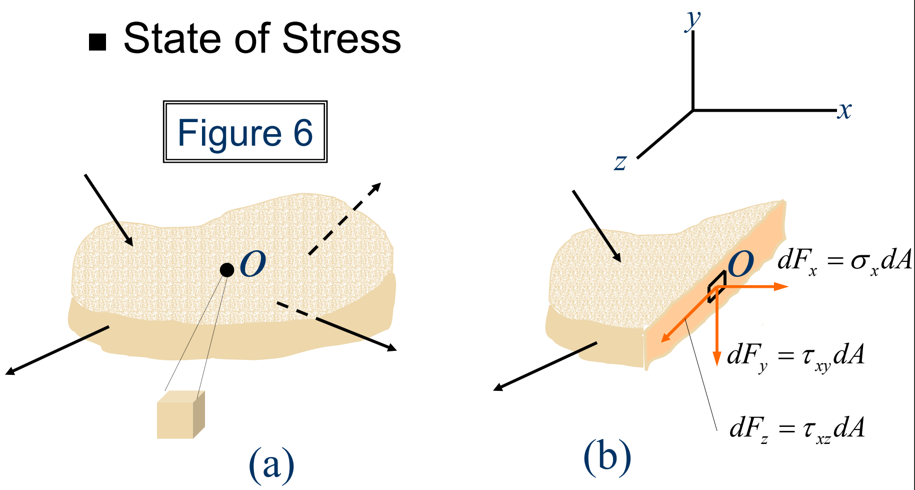
# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress
  - Stress at a point in a material body has been defined as a force per unit area.
  - But this definition is somewhat ambiguous since it depends upon what area we consider at the point.
  - To see this, consider a point *O* in the interior of the body shown in Fig. 6a.

# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress

Figure 6






# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress

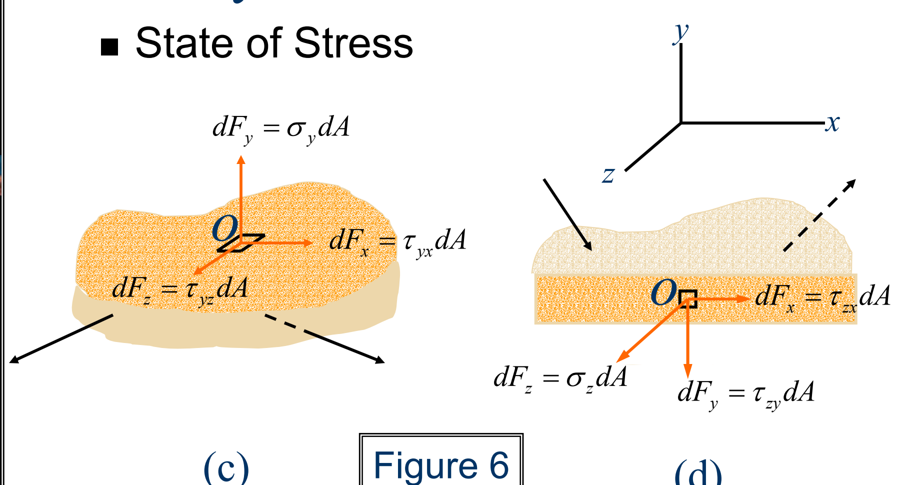



Figure 6

# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress
  - Let *x*, *y*, and *z* be the usual rectangular coordinates axes and let us pass a cutting plane through point *O* perpendicular to the *x* axis as shown in Fig. 6b.
  - If *dA* is the area, then by definition,

$$\sigma_x = \frac{dF_x}{dA} \qquad \tau_{xy} = \frac{dF_y}{dA} \qquad \tau_{xz} = \frac{dF_z}{dA} \qquad (5)$$



# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress
  - In a similar manner, let us pass a cutting plane through point *O* perpendicular to the *y* axis as shown in Fig. 6c.
  - The corresponding components of stresses can be written as

$$\sigma_y = \frac{dF_y}{dA} \qquad \tau_{yx} = \frac{dF_x}{dA} \qquad \tau_{yz} = \frac{dF_z}{dA} \qquad (6)$$

# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress
  - Finally, by passing a cutting plane perpendicular to the *z* axis as in Fig. 6d, this can result in

$$\sigma_z = \frac{dF_z}{dA} \qquad \tau_{zx} = \frac{dF_x}{dA} \qquad \tau_{zy} = \frac{dF_y}{dA} \qquad (7)$$



# Stress at a General Point in an Arbitrary Loaded Member

- State of Stress
  - Now it is quite likely that each of the nine stresses provided by Eqs. 5, 6, and 7 will have a different value.
  - Therefore, *what is the stress at a point?*
  - It seems that we have nine choices!
  - There is no such a thing as *the stress* at a point $O$, but rather there is a combination or *state of stress* at point $O$.

# Stress at a General Point in an Arbitrary Loaded Member

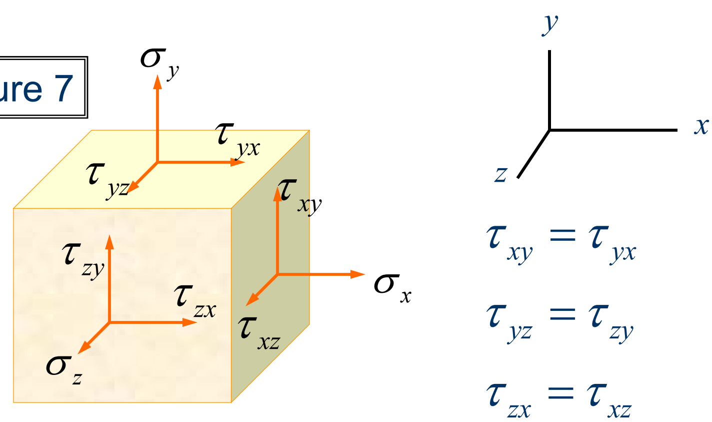
- State of Stress
  - It is convenient to depict the state of stress by the scheme of Fig. 7, in which the stresses on three mutually perpendicular planes are labeled in the manner described above.
  - The state of stress shown in the figure is called the <u>*general or triaxial*</u> *state of stress* which can exit at any interior point.



# Stress at a General Point in an Arbitrary Loaded Member

- General or Triaxial State of Stress

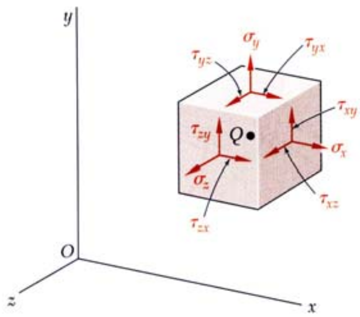
Figure 7




$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy} \qquad (7)$$
$$\tau_{zx} = \tau_{xz}$$

# Stress at a General Point in an Arbitrary Loaded Member

- General or Triaxial State of Stress




- The most general state of stress at a point may be represented by 6 components,

$\sigma_x, \sigma_y, \sigma_z$ normal stresses

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

(Note: $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)



# Stress at a General Point in an Arbitrary Loaded Member

- General or Triaxial State of Stress
  - Sign Conventions

    Normal stresses indicated by the symbol $\sigma$ and a single subscript to indicate the plane (actually the outward normal to the plane) on which he stress acts.

    Normal stresses are positive if they point in the direction of the outward normal. Thus, normal stresses are positive if tensile and negative if compressive.

# Stress at a General Point in an Arbitrary Loaded Member

- General or Triaxial State of Stress
  - Sign Conventions (cont'd)

    Shearing stresses are denoted by the symbol $\tau$ followed by two subscripts, the first subscript designates the normal to the plane on which the stress acts and the second designate the coordinate axis to which the stress is parallel.

    A positive shearing stress points in the positive direction of the coordinate axis of the second subscript if it acts on a surface with an outward normal in the positive direction.



# Stress at a General Point in an Arbitrary Loaded Member

- General or Triaxial State of Stress
  - In dealing with the states of stresses the engineer is confronted with two problems:
    1. First, how does she/he determine the state of stress of a point; that is, how does she/he calculate values for $\sigma_x$, $\sigma_y$, $\tau_{xy}$, and so on?
    2. Second, how does the maximum value of the stress (normal or shear) be determined at a point? The stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$, and so on may not be the maximum possible value.

# Two-Dimensional Stress

- Plane Stress
  – The analysis of plane stresses of a point are more simpler than the analysis of general state of stress.
  – Plane stress is considered a special case of the general state of stress of a point.
  – Two parallel faces of the small element shown in Fig. 7 and 8 are assumed to be free of stress.



# Two-Dimensional Stress

- Plane Stress

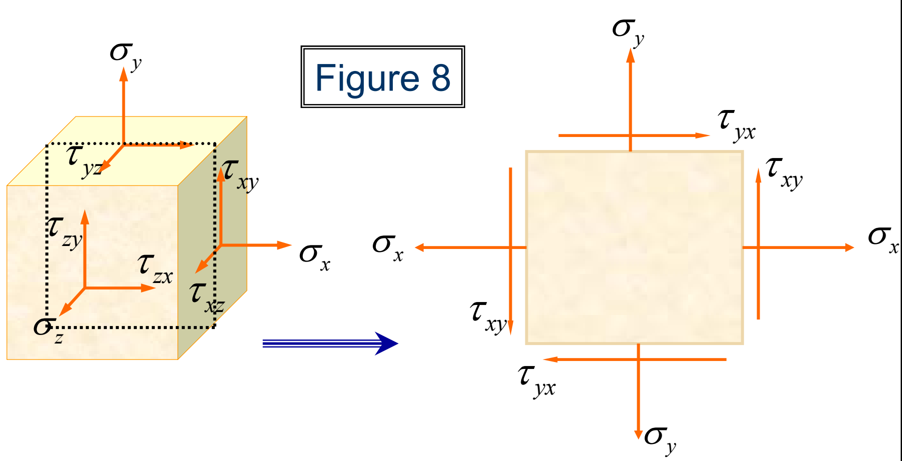
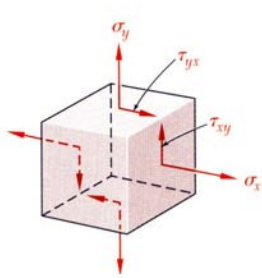



Figure 8

# Two-Dimensional Stress

- Plane Stress



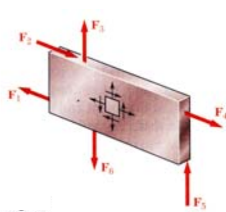

- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by

$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

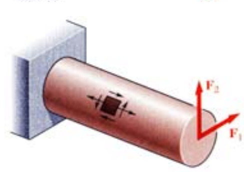


# Two-Dimensional Stress

- Plane Stress



- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.



- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

# Two-Dimensional Stress

- Plane Stress
  - For the purpose of a analysis, let these faces be perpendicular to the *z*-axis, therefore,

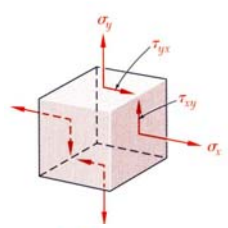
$$\sigma_z = \tau_{zx} = \tau_{zy} = 0 \qquad (8)$$

  - As was shown earlier, this also implies that

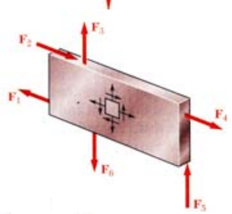
$$\tau_{xz} = \tau_{yz} = 0 \qquad (9)$$



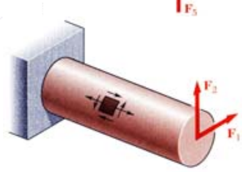
# Introduction



- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by $\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_z = \tau_{zx} = \tau_{zy} = 0.$



- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.



- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

# Two-Dimensional Stress

- Plane Stress
  - Components:
    - Normal Stress $\sigma_x$
    - Normal Stress $\sigma_y$
    - Shearing Stress $\tau_{xy}$
    - Shearing Stress $\tau_{yx}$

$$\tau_{xy} = \tau_{yx}$$

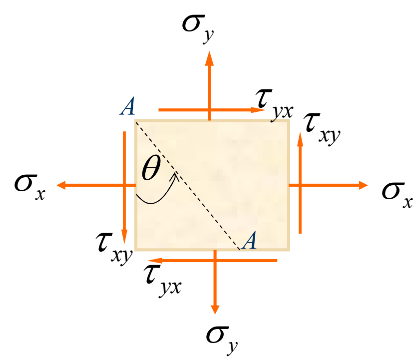



Figure 9



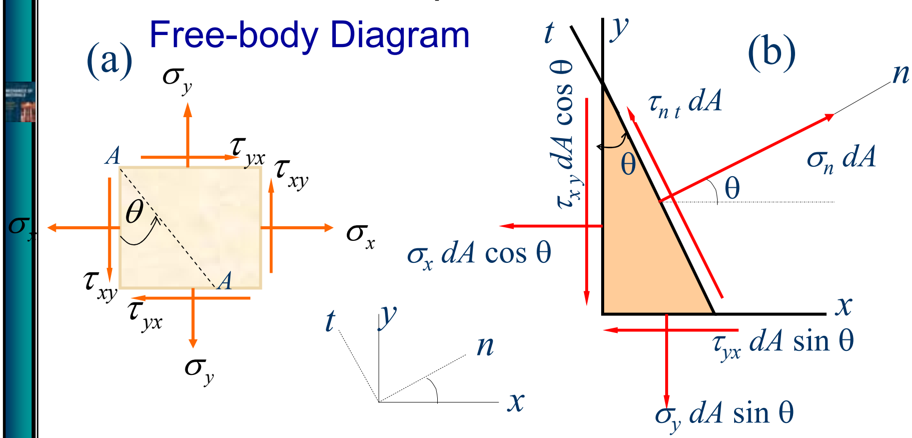
# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations
  - Equations relating the normal and shearing stresses $\sigma_n$ and $\tau_{nt}$ on an arbitrary plane (oriented at an angle $\theta$ with respect to a reference *x*-axis) through a point and the known stresses $\sigma_x$, $\sigma_y$, and $\tau_{xy} = \tau_{yx}$ on the reference planes can be developed using the free body-diagram method.

# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations

Figure 10






# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations
  - Figure 10b is a free-body diagram of wedge-shaped element in which the areas of the faces are
    - *dA* for the inclined face (plane *A-A*)
    - *dA* cos θ for the vertical face, and
    - *dA* sin θ for the horizontal face

# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations

Summing forces in the *n*-direction gives

$$+\nearrow\sum F_n = \sigma_n dA - \sigma_x(dA\cos\theta)\cos\theta - \sigma_y(dA\sin\theta)\sin\theta - \tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta = 0$$

Since $\tau_{xy} = \tau_{yx}$, therefore

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (10a)$$

In terms of the double angle,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (10b)$$

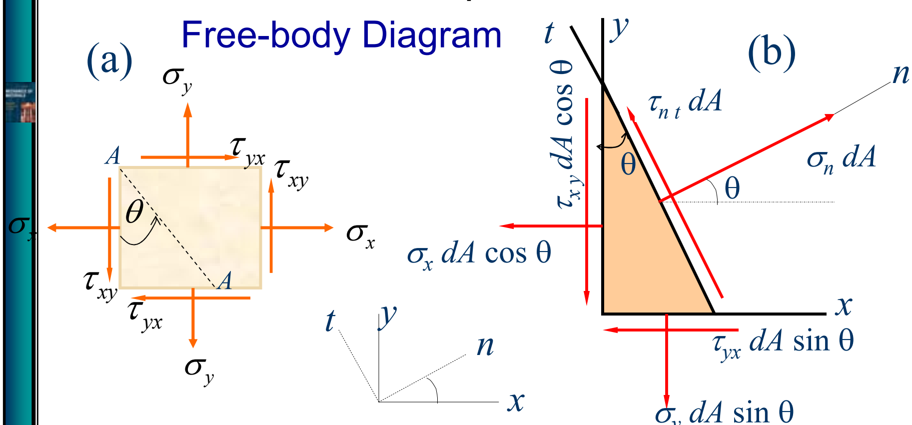


# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations

Figure 10

Free-body Diagram



# The Stress Transformation Equations for Plane Stress

- Plane Stress Equations

Summing forces in the *t*-direction gives

$$+\nwarrow \sum F_t = \tau_{nt} dA + \sigma_x (dA\cos\theta)\sin\theta - \sigma_y (dA\sin\theta)\cos\theta$$
$$- \tau_{yx}(dA\cos\theta)\cos\theta + \tau_{xy}(dA\sin\theta)\sin\theta = 0$$

Since $\tau_{xy} = \tau_{yx}$, therefore

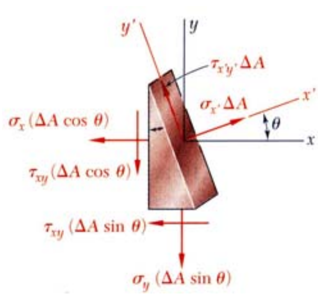
$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (11a)$$

In terms of the double angle,

$$\tau_{nt} = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (11b)$$



# Transformation of Plane Stress




- Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the *x*, *y*, and *x'* axes.

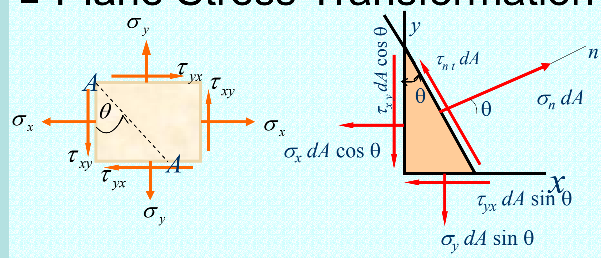
$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta$$
$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta$$

- The equations may be rewritten to yield

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

# The Stress Transformation Equations for Plane Stress

- Plane Stress Transformation Equations

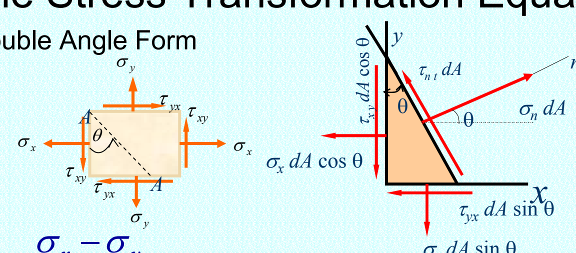



$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad (12)$$




# The Stress Transformation Equations for Plane Stress

- Plane Stress Transformation Equations
  - Double Angle Form




$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad (13)$$

# The Stress Transformation Equations for Plane Stress

- Sign Conventions
  1. Tensile normal stresses are positive; compressive normal stresses are negative.
  2. A shearing stress is positive if it points in the positive direction of the ordinate axis of the second subscript when it is acting on a surface whose outward normal is in a positive direction.



# The Stress Transformation Equations for Plane Stress

- Sign Conventions (cont'd)
  3. An angle measured counterclockwise from the reference *x*-axis is positive. Conversely, angles measured clockwise from the reference *x*-axis are negative.
  4. The *n*, *t*, *z*-axes have the same order as the *x*, *y*, *z*-axes. Both sets of axes constitute a right-hand coordinate system.

# The Stress Transformation Equations for Plane Stress

- Example 1
  - The stresses shown in the figure act at a point on the free surface of a stressed body. Determine the normal and shearing stresses at this point on the inclined plane *AB* shown in the figure.



# The Stress Transformation Equations for Plane Stress

- Example 1 (cont'd)

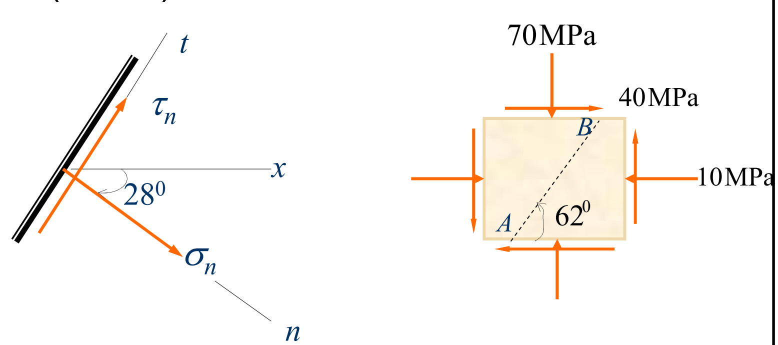


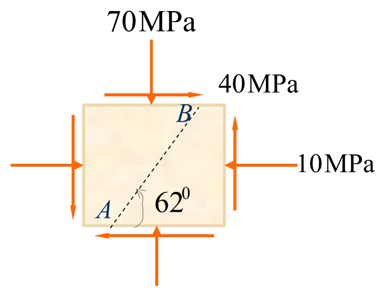

Figure 11

# The Stress Transformation Equations for Plane Stress

- Example 1 (cont'd)

We have,

$$\sigma_x = -10 \text{ MPa}$$
$$\sigma_y = -70 \text{ MPa}$$
$$\tau_{xy} = +40 \text{ MPa}$$
$$\theta = -28^0$$






# The Stress Transformation Equations for Plane Stress

- Example 1 (cont'd)

Applying Eq. 12 for the given values

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= -10\cos^2(-28) - 70\sin^2(-28) + 2(40)\sin(-28)\cos(-28)$$

$$\boxed{\sigma_n = -56.39 \text{ MPa}}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -(-10-(-70))\sin(-28)\cos(-28) + 40(\cos^2(-28) - \sin^2(-28))$$

$$\boxed{\tau_{nt} = 47.24 \text{ MPa}}$$

# The Stress Transformation Equations for Plane Stress

- Example 1 (cont'd)