LECTURE

19

Chapter 9.7 – 9.8

The McGraw-Hill Companies

Third Edition

BEAMS: DEFORMATION BY SUPERPOSITION

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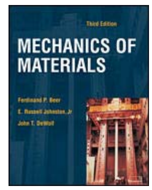

by

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ENES 220 – Mechanics of Materials

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Deflection by Superposition

- Method of Superposition
  - When a beam is subjected to several loads (see Fig. 18) at various positions along the beam, the problem of determining the slope and the deflection usually becomes quite involved and tedious.
  - This is true regardless of the method used.
  - However, many complex loading conditions are merely combinations of

Deflection by Superposition

- Method of Superposition

  relatively simple loading conditions

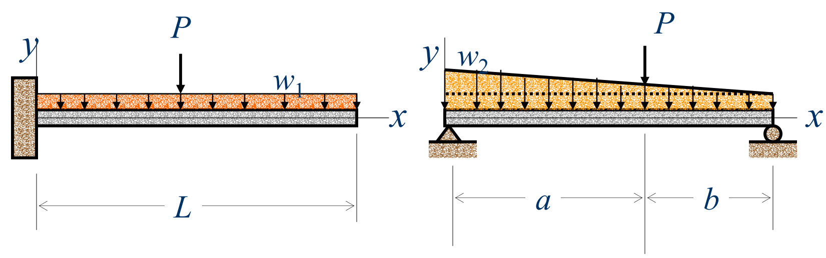


Figure 18 **(a)** **(b)**

Deflection by Superposition

- Method of Superposition
  - Assumptions:
    - The beam behaves elastically for the combined loading.
    - The beam also behaves elastically for the each of the individual loads.
    - Small deflection theory.

Deflection by Superposition

- Method of Superposition

  If it is assumed that the beam behaves elastically for the combined loading, as well as for the individual loads, the resulting final deflection of the loaded beam is simply the sum of the deflections caused by each of the individual loads.

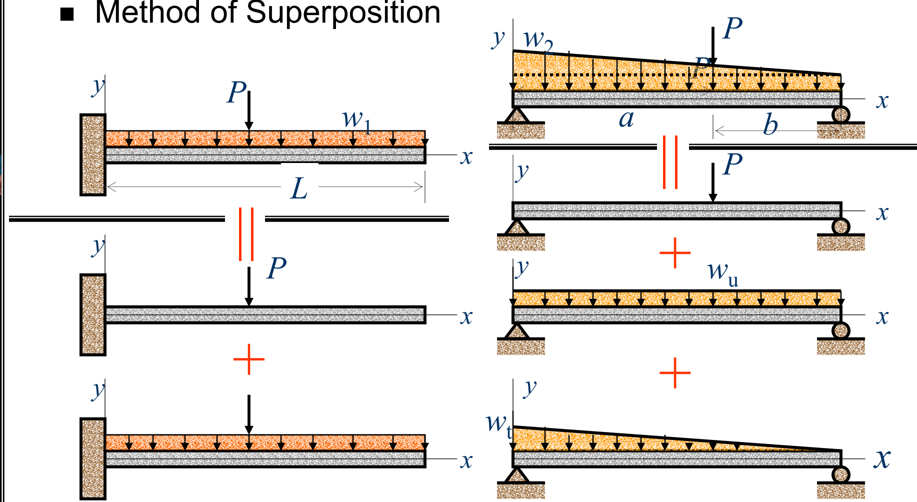
Deflection by Superposition

- Method of Superposition
  - This sum may be an algebraic one (Figure 19) or it might be a vector sum as shown in Figure 20, the type depending on whether or not the individual deflection lie in the same plane.
  - The superposition method can illustrated by various practical examples.

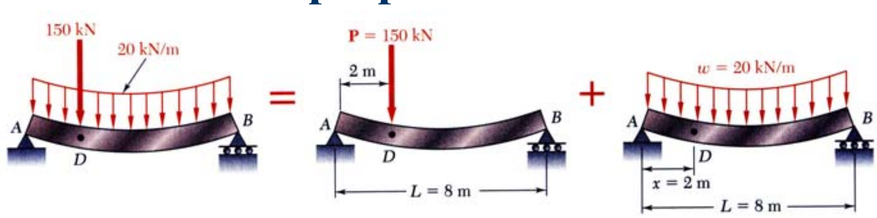
Deflection by Superposition

- Method of Superposition

Figure 19



Method of Superposition



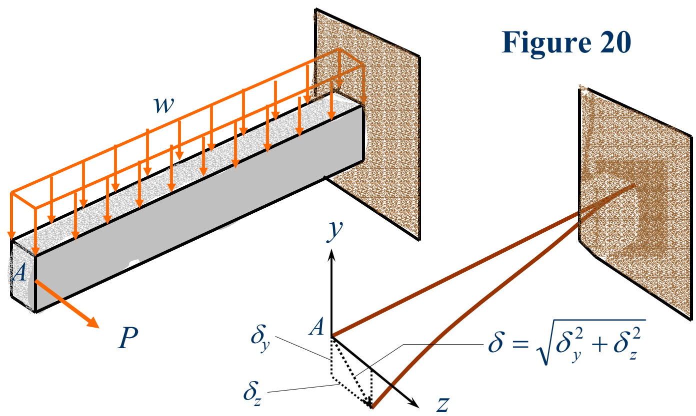
Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

Deflection by Superposition

- Method of Superposition

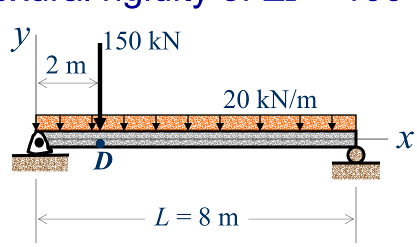
Figure 20



Deflection by Superposition

- Illustrative Example for the Use of Superposition
  – Consider the beam shown in Fig. 21, with a flexural rigidity of EI = 100 MN·m.

Figure 21



Deflection by Superposition

- Illustrative Example for the Use of Superposition
  – If we are interested on finding the slope and the deflection, say of point *D*, then we can use the superposition method to do that as illustrated in the following slides.
  – First we find the slope and deflection due the effect of each load, i.e., *w, P*, etc.

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  – The resulting final slope and deflection of point *D* of the loaded beam is simply the sum of the slopes and deflections caused by each of the individual loads as shown in Figure 22.
  – We need to find both the slope and deflection caused by the concentrated load (120 kN) and distributed load (20 kN/m)

Deflection by Superposition

- Illustrative Example for the Use of Superposition

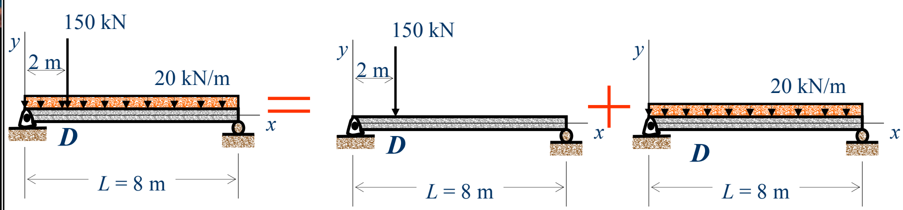
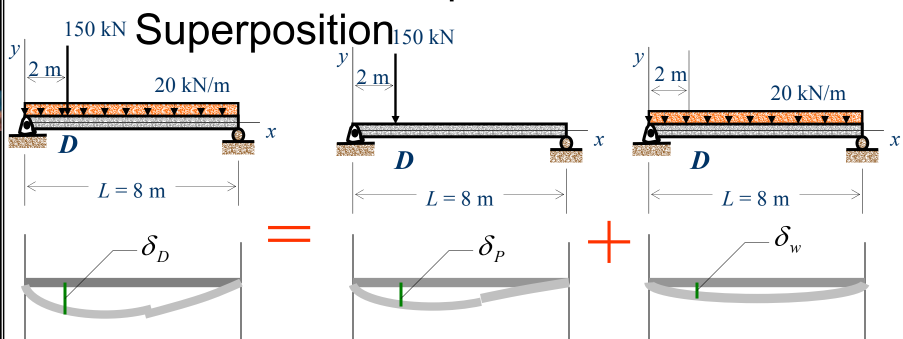


Figure 22. **Original Loading is Broken into Two Individual Loads**

Deflection by Superposition

- Illustrative Example for the Use of Superposition



$$\delta_D = \delta_{D\text{–due to } P} + \delta_{D\text{–due to } w}$$

Figure 23. **Original Deflection is Broken into Two Individual Deflections**

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  - Slope and Deflection caused by *P*
    - By either the direct integration or the singularity functions method, it can be seen that the slope and deflection (due to *P*) of point *D* of this particular loaded beam are given, respectively, as

$$(\theta_D)_P = -\frac{PL^2}{32EI} \quad \text{and} \quad (y_D)_P = \frac{3PL^3}{256EI}$$

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  - Slope and Deflection caused by *P*
    - Therefore,

$$(\theta_D)_P = -\frac{PL^2}{32EI} = \frac{150\times 10^3 (8)^2}{32(100\times 10^6)} = -0.003 \text{ rad} \qquad (25a)$$

$$(y_D)_P = -\frac{3PL^3}{256EI} = \frac{3(150\times 10^3)(8)^3}{256(100\times 10^6)} = -0.009 \text{ m} \qquad (25b)$$

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  - Slope and Deflection caused by *w*
    - By either the direct integration or the singularity functions method, it can be seen that the slope and deflection (due to *w*) of point *D* of this particular loaded beam are given, respectively, as

$$(\theta_D)_P = \frac{w}{24EI}\left(-4x^3 + 6Lx^2 - L^3\right) \qquad (26a)$$

$$(y_D)_P = \frac{w}{24EI}\left(-x^4 + 2Lx^3 - L^3x\right) \qquad (26b)$$

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  - Slope and Deflection caused by *w*
    - With *w* = 20 kN/m, *x* = 2 m, and *L* = 8 m, thus

$$(\theta_D)_P = \frac{w}{24EI}\left(-4x^3 + 6Lx^2 - L^3\right) = \frac{20\times10^3}{24\left(100\times10^6\right)}(-356) = -0.00293\ \text{rad}$$

$$(y_D)_P = \frac{w}{24EI}\left(-x^4 + 2Lx^3 - L^3x\right) = \frac{20\times10^3}{24\left(100\times10^6\right)}(-912) = -0.0076\ \text{m}$$

Deflection by Superposition

- Illustrative Example for the Use of Superposition
  - Combining the slopes and deflections produced by the concentrated (*P*) and distributed (*w*) loads, the results are

$$\theta_D = (\theta_D)_P + (\theta_D)_w = -0.003 - 0.00293 = \boxed{-0.00593 \text{ rad}}$$

$$y_D = (y_D)_P + (y_D)_w = -0.009 - 0.0076 = 0.0166 \text{ m} = \boxed{16.6 \text{ mm}}$$

Deflection by Superposition

- Illustrative Example for the Use of Superposition

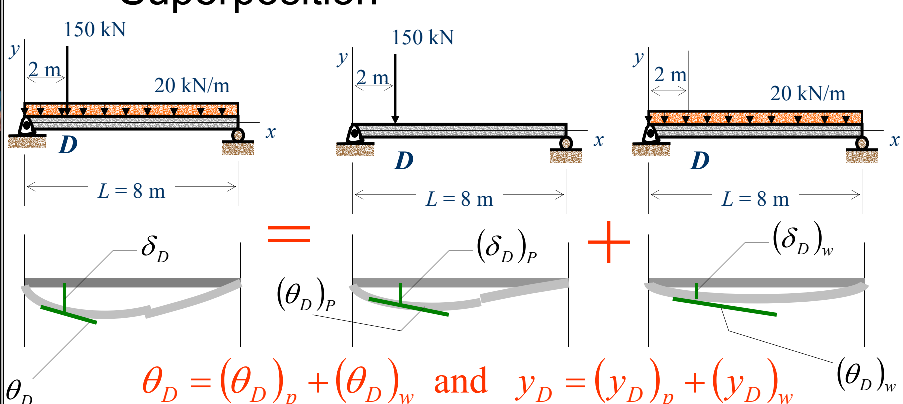


Figure 24. **Total Slope and Deflection of Point *D***

Deflection by Superposition

- General Procedure of Superposition
  - It is evident from the last results that the slope or deflection of a beam is the sum of the slopes or deflections produced by the individual loads.
  - Once the slopes or deflections produced by a few typical individual loads have been determined by one of the methods already

Deflection by Superposition

- General Procedure of Superposition
  - Presented, the superposition method provides a means of quickly solving a wide range of more complicated problems by various combinations of known results.
  - As more data become available, yet a wider range of problems can be solved by the method of superposition.

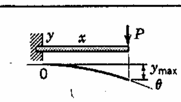
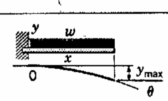
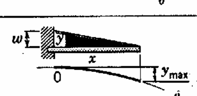
Deflection by Superposition

- Slope and Deflection Tables
  - To facilitate the task of practicing engineers, most structural and mechanical handbooks include tables giving the deflections and slopes of beams for various loadings and types of support.
  - Such a table can be found in the textbook (Table B19) and provided herein in the next few viewgraphs (Table 1 and 2).

Deflection by Superposition

- Slopes and Deflection Tables

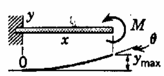
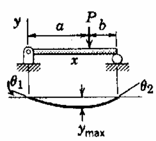
Table 1a

Case	Load and Support (Length L)	Slope at End (+∠)	Maximum Deflection (+ upward)
1		$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\frac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\frac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{30EI}$ at $x = L$

Deflection by Superposition

- Slopes and Deflection Tables

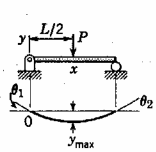
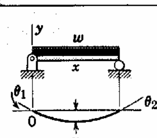
Table 1b

Case	Load and Support (Length L)	Slope at End ($+\measuredangle$)	Maximum Deflection (+ upward)
4		$\theta = +\frac{ML}{EI}$ at $x = L$	$y_{\max} = +\frac{ML^2}{2EI}$ at $x = L$
5	 $b > a$	$\theta_1 = -\frac{Pb(L^2 - b^2)}{6LEI}$ at $x = 0$ $\theta_2 = +\frac{Pa(L^2 - a^2)}{6LEI}$ at $x = L$	$y_{\max} = -\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}\ LEI}$ at $x = \sqrt{(L^2-b^2)/3}$ y_{center} not max $= -\frac{Pb(3L^2 - 4b^2)}{48EI}$

Deflection by Superposition

- Slopes and Deflection Tables

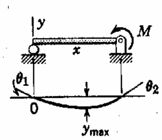
Table 1c

Case	Load and Support (Length L)	Slope at End ($+\measuredangle$)	Maximum Deflection (+ upward)
6		$\theta_1 = -\frac{PL^2}{16EI}$ at $x = 0$ $\theta_2 = +\frac{PL^2}{16EI}$ at $x = L$	$y_{\max} = -\frac{PL^2}{48EI}$ at $x = L/2$
7		$\theta_1 = -\frac{wL^3}{24EI}$ at $x = 0$ $\theta_2 = +\frac{wL^3}{24EI}$ at $x = L$	$y_{\max} = -\frac{5wL^4}{384EI}$ at $x = L/2$

Deflection by Superposition

- Slopes and Deflection Tables

Table 1d

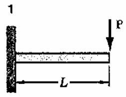
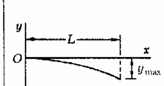
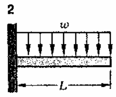
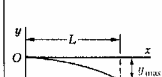
Case	Load and Support (Length L)	Slope at End ($+\measuredangle$)	Maximum Deflection (+ upward)
8		$\theta_1 = -\frac{ML}{6EI}$ at $x = 0$ $\theta_2 = +\frac{ML}{3EI}$ at $x = L$	$y_{\max} = -\frac{ML^2}{9\sqrt{3}EI}$ at $x = L/\sqrt{3}$ $y_{\text{center}} = -\frac{ML^2}{16EI}$ not max

Deflection by Superposition

- Slopes and Deflection Tables

Table 2a

Appendix D. Beam Deflections and Slopes (Beer and Johnston 1992)

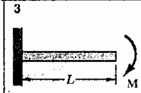
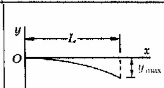
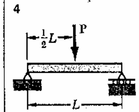
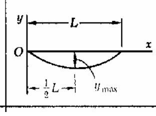
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$

Deflection by Superposition

- Slopes and Deflection Tables

Table 2b

Appendix D. Beam Deflections and Slopes (Beer and Johnston 1992)

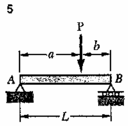
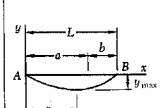
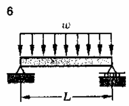
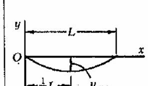
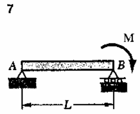
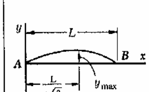
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$

Deflection by Superposition

- Slopes and Deflection Tables

Table 2c

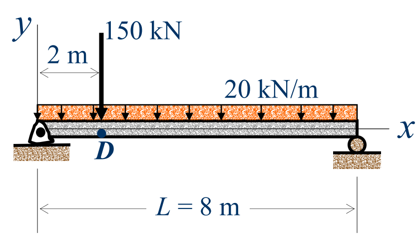
(Beer and Johnston 1992) Appendix D. Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
5 		For $a > b$: $-\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2-a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2-b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$

Deflection by Superposition

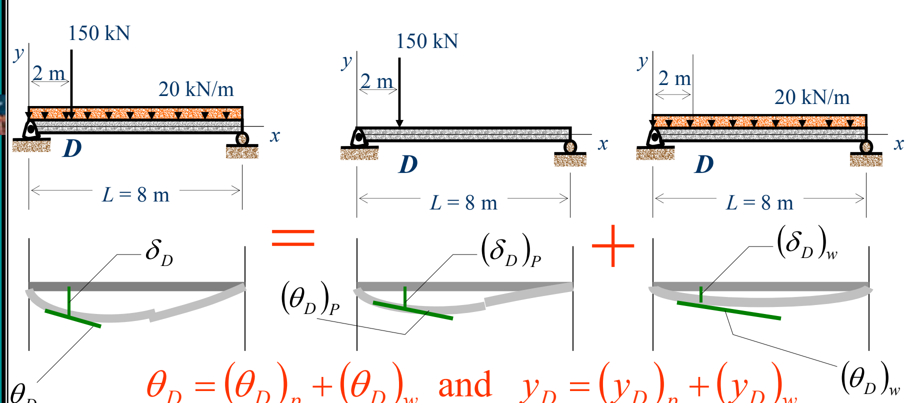
- Use of Slopes and Deflection Tables
  - Notice that the slope and deflection of the beam of Figures 21 and 24 (repeated here) of the illustrative example could have been determined from the table (Table 1)

Figure 21



Deflection by Superposition

- Use of Slopes and Deflection Tables



$$\theta_D = (\theta_D)_p + (\theta_D)_w \quad \text{and} \quad y_D = (y_D)_p + (y_D)_w$$

Figure 24. Total Slope and Deflection of Point *D*

Deflection by Superposition

- Use of Slopes and Deflection Tables
  - Indeed, given the information given under cases 5 and 6 of Tables 2c, the slope and deflection for any value $x \leq L/4$ could have been expressed analytically.
  - Taking the derivative of the expression obtained in this way, would have yielded the slope of the beam over the same interval.

Deflection by Superposition

- Use of Slopes and Deflection Tables
  - The slope at both ends of the beam may be obtained by simply adding the corresponding values given in the table.
  - However, the maximum deflection of the beam of Fig. 21 *cannot* be obtained by adding the maximum deflections of cases 5 and 6 (Table 2c), since these deflections occur at different points of the beam.

Deflection by Superposition

- Use of Slopes and Deflection Tables
  - Applying case 5 on the illustrative example to find both the slope and deflection of point *D* of the beam (Fig. 21), yields

$$(y_D)_P = \frac{Pb}{6EIL}\left[x^3 - \left(L^2 - b^2\right)x\right] = \frac{150\times10^3(6)}{6\left(100\times10^6\right)(8)}\left[2^3 - \left(8^2 - 6^2\right)(2)\right] = -0.009 \text{ m}$$

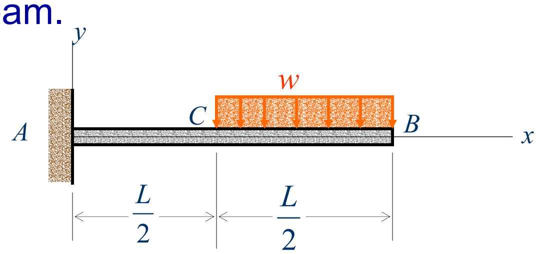
$$(\theta_D)_P = \frac{dy}{dx} = \frac{Pb}{6EIL}\left[3x^2 - \left(L^2 - b^2\right)\right] = \frac{150\times10^3(6)}{6\left(100\times10^6\right)(8)}\left[3(2)^2 - \left(8^2 - 6^2\right)\right] = -0.003 \text{ rad}$$

  - These values confirm the results obtained using Eq. 25 of the integration method.

Deflection by Superposition

- Example 6

  Use the method of superposition to find the slope and deflection at point *B* of the beam.



Deflection by Superposition

- Example 6 (cont'd)

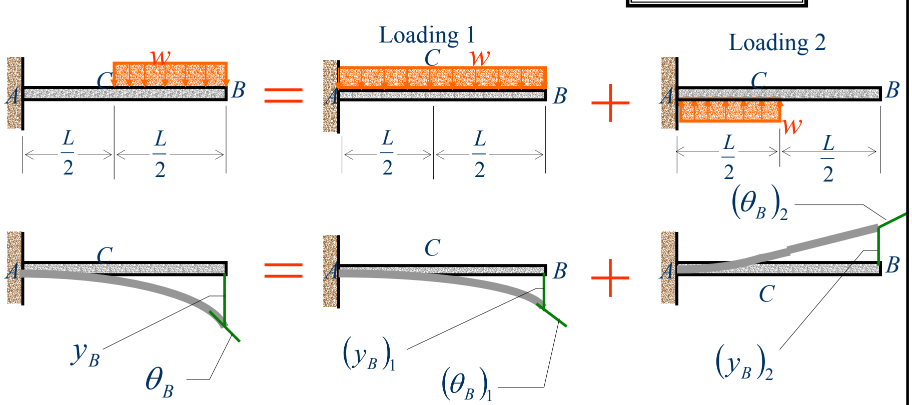
  The given loading can be obtained by superposing the loadings shown in the following "picture equation" (Fig. 25). The beam *AB* is, of course, the same in each part of the figure.

  For each the loadings 1 and 2, the slope and deflection at *B* can be determined by using the Tables 1 or 2. (Textbook Table B-19)

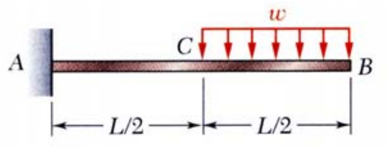
Deflection by Superposition

- Example 6 (cont'd)

Figure 25



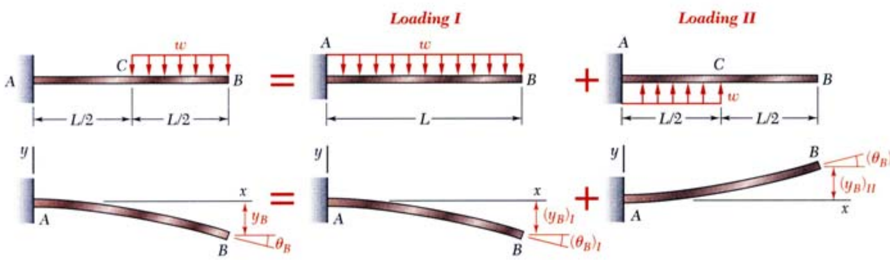
Problem 6 (cont'd)



For the beam and loading shown, determine the slope and deflection at point *B*.

SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



Deflection by Superposition

- Example 6 (cont'd)

Loading 1:

- From Table 1a or Table 2a (also Table B-19 of the textbook),

$$(\theta_B)_1 = -\frac{wL^3}{6EI} \quad \text{and} \quad (y_B)_1 = -\frac{wL^4}{8EI} \qquad (27a)$$

Loading 2:

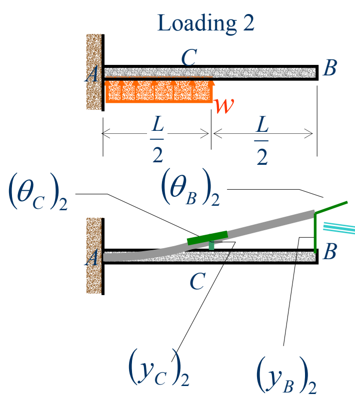
- From the same tables:

$$(\theta_C)_2 = +\frac{w(L/2)^3}{6EI} = +\frac{wL^3}{48EI} \quad \text{and} \quad (y_C)_2 = +\frac{w(L/2)^4}{8EI} = \frac{wL^4}{128EI} \qquad (27b)$$

Deflection by Superposition

- Example 6 (cont'd)

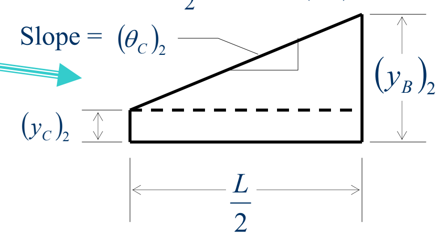
Figure 26



In portion *CB*, the bending moment for loading 2 is zero, thus the elastic curve is a straight line:

$$(\theta_B)_2 = (\theta_C)_2 = +\frac{wL^3}{48EI} \qquad (28)$$

$$(y_B)_2 = (y_C)_2 + \frac{L}{2}(\theta_C)_2 \qquad (29)$$



Deflection by Superposition

- Example 6 (cont'd)

Total slope and deflection:

- Slope of Point *B*:

$$\theta_B = (\theta_B)_1 + (\theta_B)_2 = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} = \boxed{-\frac{7wL^3}{48EI}}$$

- Deflection of Point *B*:

$$(y_B)_2 = (y_C)_2 + \frac{L}{2}(\theta_C)_2 = \frac{wL^4}{128EI} + \frac{L}{2}\left(\frac{wL^3}{48EI}\right) = +\frac{7wL^4}{384EI}$$

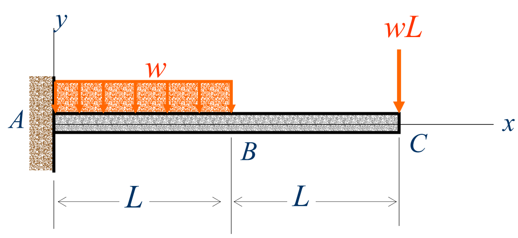
$$y_B = (y_B)_1 + (y_B)_2 = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} = \boxed{-\frac{41wL^4}{384EI}}$$

Deflection by Superposition

- Example 7

Use the method of superposition, determine the deflection at the free end of the cantilever beam shown in Fig. 27 in terms of *w*, *L*, *E*, and *I*.

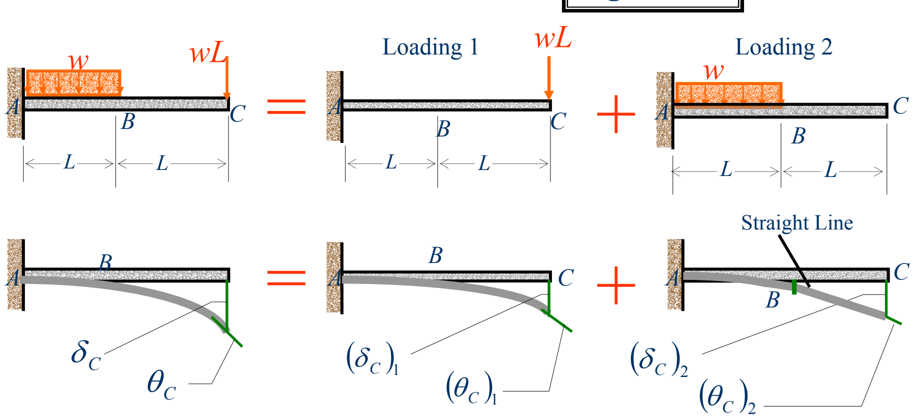
Figure 27



Deflection by Superposition

- Example 7 (cont'd)

Figure 28



Deflection by Superposition

- Example 7 (cont'd)

  Using the solutions listed in Table 1a. Cases 1 and 2 (Textbook Table B-19) with *P* = *wL*

$$\delta_C = (\delta_C)_1 + (\delta_C)_2 = (\delta_C)_1 + (\delta_B)_2 + L(\theta_B)_2$$

$$= -\frac{P(2L)^3}{3EI} + \left[-\frac{wL^4}{8EI} - L\left(\frac{wL^3}{6EI}\right)\right]$$

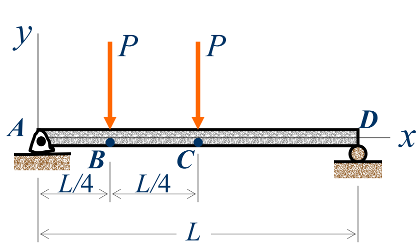
$$= -\frac{wL(2L)^3}{3EI} + \left[-\frac{wL^4}{8EI} - L\left(\frac{wL^3}{6EI}\right)\right] = \boxed{-\frac{71wL^4}{24EI}}$$

Deflection by Superposition

- Example 8

  For the simply supported beam of Fig. 29, use the method of superposition to determine the total deflection at point *C* in terms of *P*, *L*, *E*, and *I*.

Figure 29



Deflection by Superposition

- Example 8 (cont'd)

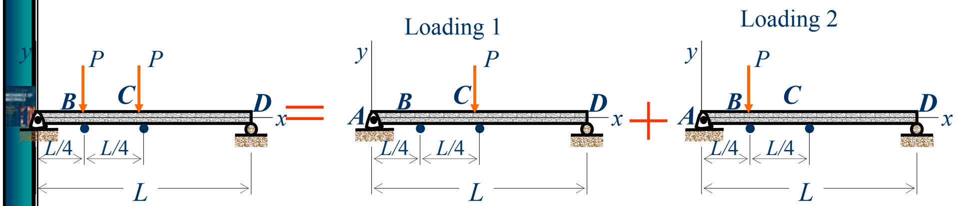


Figure 30

From Table 1c (Text B-19)
Case 6

$$x = L/2$$

$$y_{\text{center}} = -\frac{PL^3}{48EI}$$

From Table 1b (Text B-19)
Case 5

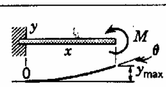
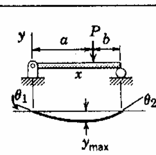
$$a = 3L/4,\ b = L/4$$

$$y_{\text{center}} = -\frac{Pb(3L^2 - 4b^2)}{48EI}$$

Deflection by Superposition

- Example 8 (cont'd)

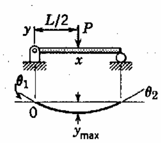
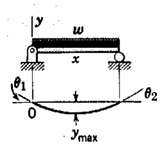
Table 1b

Case	Load and Support (Length L)	Slope at End (+⊿)	Maximum Deflection (+ upward)
4		$\theta = +\frac{ML}{EI}$ at $x = L$	$y_{max} = +\frac{ML^2}{2EI}$ at $x = L$
5	 $b > a$	$\theta_1 = -\frac{Pb(L^2 - b^2)}{6LEI}$ at $x = 0$; $\theta_2 = +\frac{Pa(L^2 - a^2)}{6LEI}$ at $x = L$	$y_{max} = -\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}\ LEI}$ at $x = \sqrt{(L^2-b^2)/3}$; $y_{\text{center not max}} = -\frac{Pb(3L^2 - 4b^2)}{48EI}$

Deflection by Superposition

- Example 8 (cont'd)

Table 1c

Case	Load and Support (Length L)	Slope at End (+⊿)	Maximum Deflection (+ upward)
6		$\theta_1 = -\frac{PL^2}{16EI}$ at $x = 0$; $\theta_2 = +\frac{PL^2}{16EI}$ at $x = L$	$y_{max} = -\frac{PL^2}{48EI}$ at $x = L/2$
7		$\theta_1 = -\frac{wL^3}{24EI}$ at $x = 0$; $\theta_2 = +\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{5wL^4}{384EI}$ at $x = L/2$

Deflection by Superposition

- Example 8 (cont'd)

Deflection due to Loading 1:

$$(y_C)_1 = \frac{PL^3}{48EI}$$

Deflection due to Loading 2:

$$(y_C)_2 = -\frac{Pb\left(3L^2 - 4b^2\right)}{48EI} = -\frac{P(L/4)\left[3L^2 - 4(L/4)^2\right]}{48EI} = -\frac{11PL^3}{768EI}$$

Therefore, total deflection of point C

$$y_C = (y_C)_1 + (y_C)_2 = -\frac{PL^3}{48EI} - \frac{11PL^3}{768EI} = \boxed{-\frac{9PL^3}{256EI}}$$

Deflection by Superposition

- Example 9

Using the method of superposition, find the deflection at a point midway between the supports of the beam shown in the figure in terms of *w*, *L*, *E*, and *I*.

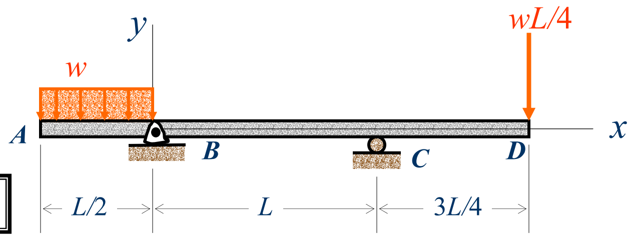


Figure 31

Deflection by Superposition

- Example 9 (cont'd)

The deflection at a point midway between the supports can be determined by considering the beam shown in Fig. 32.

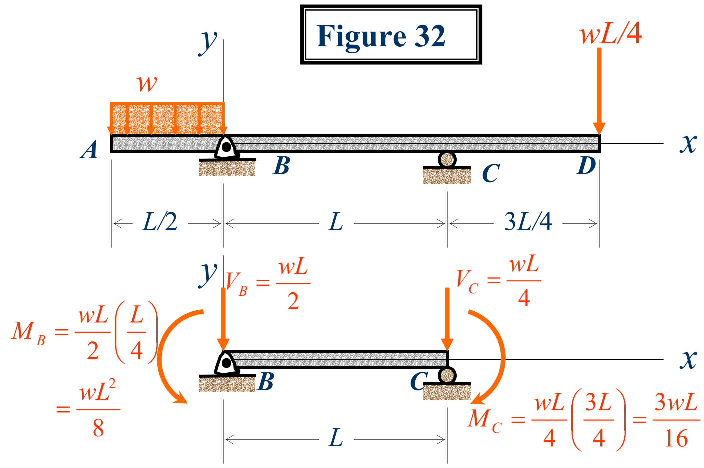
Note that since the shear forces V_B and V_C do not contribute to the deflection at any point in span *BC*, the mid-span deflection can be expressed as

$$\delta_{\text{mid}} = \delta_{M_B} + \delta_{M_C} \qquad (28)$$

Deflection by Superposition

- Example 9 (cont'd)

Figure 32



Deflection by Superposition

- Example 9 (cont'd)

  Using the solutions listed in Table 1, Table 2, or Table B-19 of the textbook with $M_B = wL^2/8$ and $M_C = 3wL^2/16$

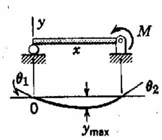
$$\delta_{\text{mid}} = \delta_{M_B} + \delta_{M_C} = \frac{\left(wL^2/8\right)\left(L^2\right)}{16EI} + \frac{\left(3wL^2/16\right)\left(L^2\right)}{16EI}$$

$$= \boxed{\frac{5wL^4}{256EI}}$$

Deflection by Superposition

- Example 9 (cont'd)

Table 1d

Case	Load and Support (Length L)	Slope at End (+⊿)	Maximum Deflection (+ upward)
8		$\theta_1 = -\dfrac{ML}{6EI}$ at $x = 0$ $\theta_2 = +\dfrac{ML}{3EI}$ at $x = L$	$y_{max} = -\dfrac{ML^2}{9\sqrt{3}EI}$ at $x = L/\sqrt{3}$ $y_{\substack{center \\ \text{not max}}} = -\dfrac{ML^2}{16EI}$

Deflection by Superposition

- Example 10

For the beam in Fig. 33, determine the flexural stress at point *A* and the deflection of the left-hand end.

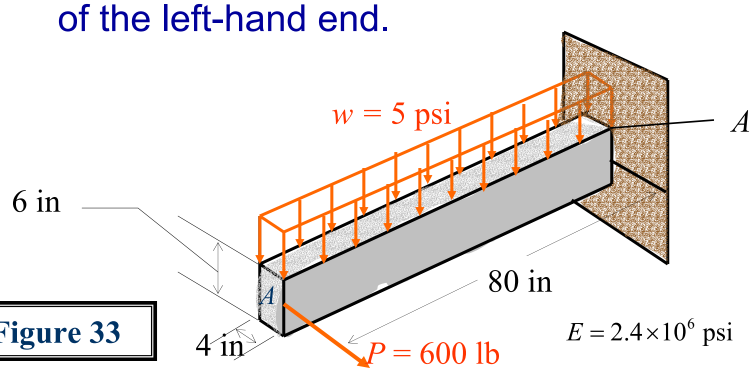


Figure 33

Deflection by Superposition

- Example 10 (cont'd)

  The stress at point *A* is a combination of compressive flexural stress due to the concentrated load and a tensile flexural stress due to the distributed load, hence,

$$\sigma_A = \frac{M_z(3)}{I_z} - \frac{M_y(2)}{I_y} = \frac{\left[5(4)(80)^2/2\right](3)}{4(6)^3/12} - \frac{600(80)(2)}{6(4)^3/12}$$

$$= 2{,}666.7 + 3000.0 = \boxed{-333.3 \text{ psi (compression)}}$$

Deflection by Superposition

- Example 10 (cont'd)

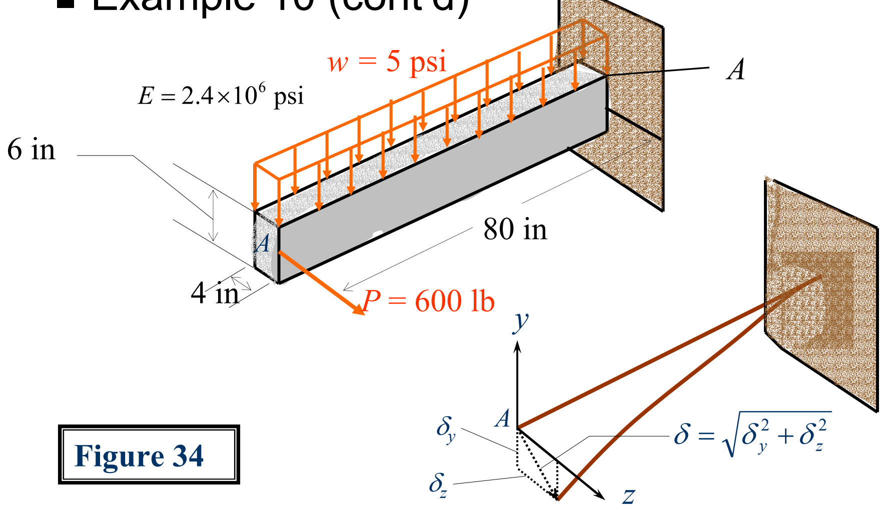


Figure 34

Deflection by Superposition

- Example 10 (cont'd)

  The deflection at the end of a cantilever beam with uniformly distributed load is given by (see Table 1a, case 2)

$$y_0 = \frac{wL^4}{8EI_z} = \frac{5(4)(80)^4}{8(2.4\times10^6)\left[4(6)^3/12\right]} = 0.5926 \text{ in}$$

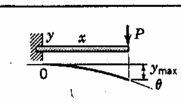
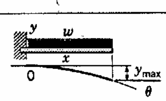
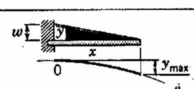
  and with concentrated load at the end is given by (see Table 1a, case 1)

$$z_0 = \frac{PL^3}{3EI} = \frac{600(80)^3}{3(2.4\times10^6)\left[6(4)^3/12\right]} = 1.3333 \text{ in}$$

Deflection by Superposition

- Example 10 (cont'd)

Table 1a

Case	Load and Support (Length L)	Slope at End (+∠)	Maximum Deflection (+ upward)
1		$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\frac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\frac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{30EI}$ at $x = L$

Deflection by Superposition

- Example 10 (cont'd)

  Superimposing the results for the deflections due to the concentrated and distributed loads, the deflection at the free end is the vector sum:

$$\delta = \sqrt{y_0^2 + z_0^2} = \sqrt{(0.5626)^2 + (1.3333)^2}$$

$$= \boxed{1.447 \text{ in}}$$

Statically Indeterminate Transversely Loaded Beams

- The Superposition Method
  - The concept of the *superposition*, which states that a slope or deflection due to several loads is the algebraic sum of the slopes or deflections to each individual loads acting alone can be applied to statically indeterminate beams.
  - The superposition can provide the additional equations needed in the analysis.

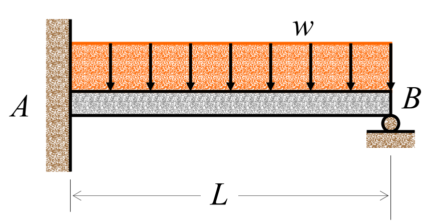
Statically Indeterminate Transversely Loaded Beams

- The Superposition Method
  - Procedure
    - Selected restraints are removed and replaced by unknown loads, e.g., forces and couples.
    - Sketching of the deformation (deflection) diagrams corresponding to individual loads (both known and unknown).
    - Adding up algebraically the individual of components of slopes or deflections to produce the known configuration.

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition

  Determine the reactions at the supports for the simply supported cantilever beam (Fig.35) presented earlier for the integration method.



Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)
  - First consider the reaction at *B* as redundant and release the beam from the support (remove restraint).
  - The reaction R_B is now considered as an unknown load (see Fig. 39) and will be determined from the condition that the deflection at *B* must be zero.

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

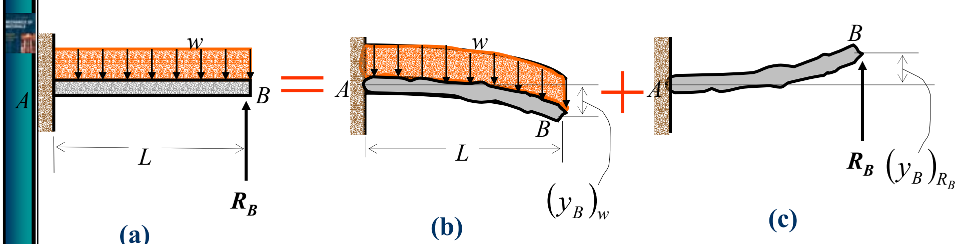
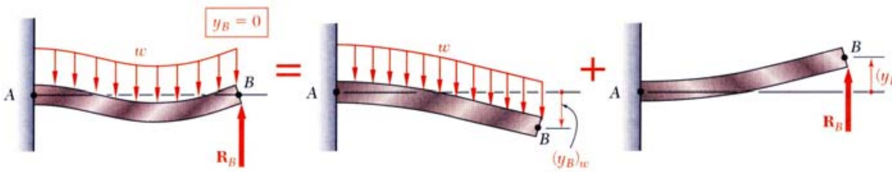


Figure 39. Original Loading is Broken into Two Loads

Application of Superposition to Statically Indeterminate Beams



- Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.
- Designate one of the reactions as redundant and eliminate or modify the support.
- Determine the beam deformation without the redundant support.
- Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

In reference to Table 1a cases 1 and 2 (Table B19 of Textbook):

$$(y_B)_{R_B} = +\frac{R_B L^3}{3EI} \quad \text{and} \quad (y_B)_w = -\frac{wL^4}{8EI} \qquad (37)$$

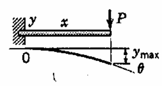
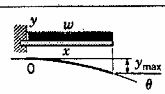
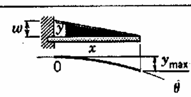
The deflection at *B* in the original structural configuration must equal to zero, that is

$$y_B = (y_B)_{R_A} + (y_B)_w = 0 \qquad (38)$$

Statically Indeterminate Transversely Loaded Beams

- Slopes and Deflection Tables

Table 1a

Case	Load and Support (Length L)	Slope at End (+⊿)	Maximum Deflection (+ upward)
1		$\theta = -\dfrac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\dfrac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\dfrac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\dfrac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\dfrac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\dfrac{wL^4}{30EI}$ at $x = L$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

Substituting Eq. 37 into Eq. 38, gives

$$+\frac{R_B L^3}{3EI} - \frac{wL^4}{8EI} = 0 \qquad (39)$$

Solving for R_B, the result is

$$\boxed{R_B = +\frac{3}{8}wL} \qquad (40)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

  From the free-body diagram for entire beam (Figure 40), the equations of equilibrium are used to find the rest of the reactions.

$$+\uparrow \sum F_y = 0;\ R_{A_y} + R_B - wL = 0$$

$$\therefore R_{A_y} = wL - R_B \qquad (41)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

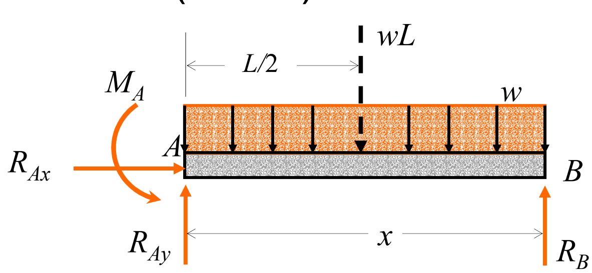


Figure 40. Free-body Diagram for the Entire Beam

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

But $R_B = \frac{3}{8}wL$ from Eq. 40, therefore

$$R_A = wL - \frac{3}{8}wL = \boxed{\frac{5}{8}wL} \qquad (42)$$

$$+\circlearrowleft \sum M_A = 0; -M_A - R_B L + (wL)\frac{L}{2} = 0$$

$$\therefore M_A = -R_B L + \frac{1}{2}wL^2 = \left(\frac{3}{8}wL\right)L - \frac{1}{2}wL^2$$

$$= \boxed{\frac{1}{8}wL^2} \qquad (43)$$

Statically Indeterminate Transversely Loaded Beams

- Illustrative Example using Superposition Method (cont'd)

From Eqs.40, 42, and 43,

$$\boxed{R_{A_y} = \frac{5}{8}wL}$$

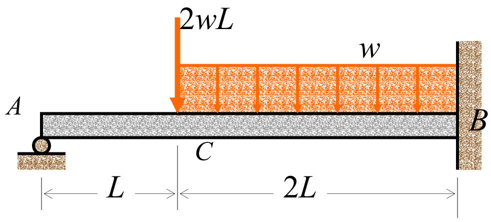
$$\boxed{M_A = \frac{1}{8}wL^2} \qquad \boxed{R_B = \frac{3}{8}wL}$$

Which confirms the results found by using the integration method.

Statically Indeterminate Transversely Loaded Beams

- Example 12

  A beam is loaded and supported as shown in the figure. Determine (a) the reaction at supports *A* and *B* in terms of *w* and *L*, and (b) the deflection at the left end of the distributed load in terms of *w*, *L*, *E*, and *I*.

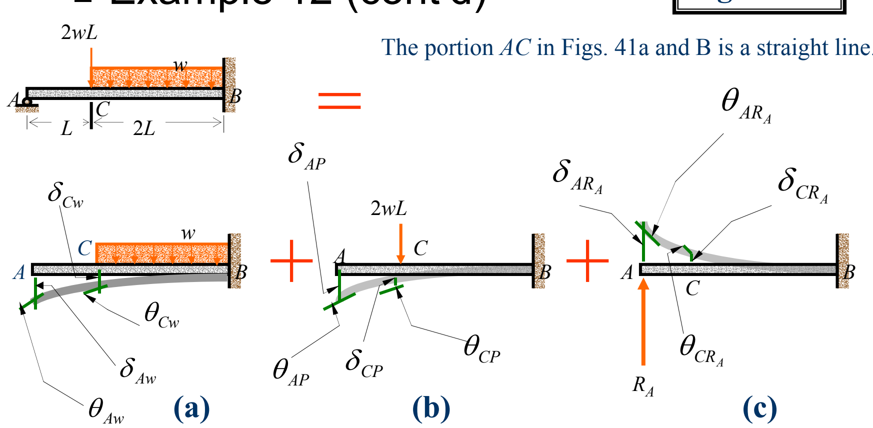


Statically Indeterminate Transversely Loaded Beams

- Example 12 (cont'd)

Figure 41

The portion *AC* in Figs. 41a and B is a straight line.



Statically Indeterminate Transversely Loaded Beams

- Example 12 (cont'd)

  Note that the portion *AC* of the beam in Figs. 41a and 41b is a straight line, therefore

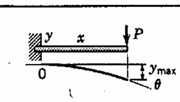
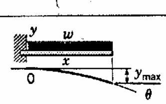
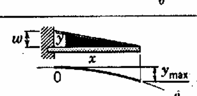
  (a) Using the solution listed in Table 1a with $P = 2wL$

$$\delta_A = \delta_{Cw} + \theta_{Cw}(L) + \delta_{CP} + \theta_{CP}(L) + \delta_{AR_A} = 0 \qquad (44)$$

$$-\frac{w(2L)^4}{8EI} - \frac{w(2L)^3}{6EI}(L) - \frac{2wL(2L)^3}{3EI} - \frac{2wL(2L)^2}{2EI}(L) + \frac{R_A(3L)^3}{3EI} = 0$$

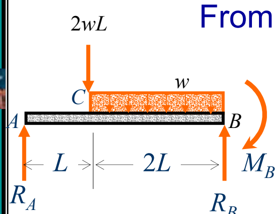
Statically Indeterminate Transversely Loaded Beams

- Slopes and Deflection Tables **Table 1a**

Case	Load and Support (Length L)	Slope at End ($+\measuredangle$)	Maximum Deflection (+ upward)
1		$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$y_{max} = -\frac{PL^3}{3EI}$ at $x = L$
2		$\theta = -\frac{wL^3}{6EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{8EI}$ at $x = L$
3		$\theta = -\frac{wL^3}{24EI}$ at $x = L$	$y_{max} = -\frac{wL^4}{30EI}$ at $x = L$

Statically Indeterminate Transversely Loaded Beams

- Example 12 (cont'd)



FBD

From which (Eq. 44),

$$R_A = +\frac{38wL}{27}$$

Equilibrium equations give

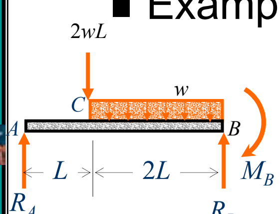
$$+\uparrow\sum F_y = 0;\ R_A - P - w(2L) + R_B = 0$$

$$\frac{38wL}{3EI} - 2wL - 2wL + R_B = 0$$

$$\therefore R_B = \boxed{\frac{70wL}{27}}$$

Statically Indeterminate Transversely Loaded Beams

- Example 12 (cont'd)



FBD

$$+\ \sum M_B = 0;\ R_A(3L) - P(2L) - w(2L)(L) + M_B = 0$$

$$\frac{38wL}{27}(3L) - 4wL^2 - 2wL^2 + M_B = 0$$

$$\therefore M_B = \boxed{\frac{16wL^2}{9}}$$

(b) Deflection at left end of distributed load (at C):

$$R_C = R_A = \frac{38wL}{27}$$

$$M_C = R_A(L) = \frac{38wL}{27}(L) = \frac{38wL^2}{27}$$

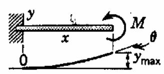
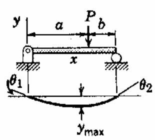
$$\delta_C = \delta_{CR_C} + \delta_{CM_C} + \delta_{CP} + \delta_{Cw}$$

$$\frac{(38wL/27)(2L)^3}{3EI} + \frac{(38wL^2/27)(2L)^2}{2EI} - \frac{2wL(2L)^3}{3EI} - \frac{w(2L)^4}{8EI} = \boxed{-\frac{62wL^4}{81EI}}$$

Statically Indeterminate Transversely Loaded Beams

- Slopes and Deflection Tables

Table 1b

Case	Load and Support (Length L)	Slope at End ($+\measuredangle$)	Maximum Deflection (+ upward)
4		$\theta = +\dfrac{ML}{EI}$ at $x = L$	$y_{max} = +\dfrac{ML^2}{2EI}$ at $x = L$
5	 $b > a$	$\theta_1 = -\dfrac{Pb(L^2 - b^2)}{6LEI}$ at $x = 0$ $\theta_2 = +\dfrac{Pa(L^2 - a^2)}{6LEI}$ at $x = L$	$y_{max} = -\dfrac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}\ LEI}$ at $x = \sqrt{(L^2-b^2)/3}$ y_{center} not max $= -\dfrac{Pb(3L^2 - 4b^2)}{48EI}$