LECTURE

17

Chapter 9.5 – 9.6

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BEAMS: DEFORMATION BY SINGULARITY FUNCTIONS

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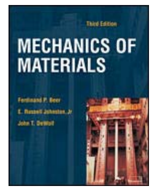

by

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ENES 220 – Mechanics of Materials

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Singularity Functions

- Introduction
  - The integration method discussed earlier becomes tedious and time-consuming when several intervals and several sets of matching conditions are needed.
  - We noticed from solving deflection problems by the integration method that the shear and moment could only rarely described by a single analytical function.

Singularity Functions

- Introduction
  - For example the cantilever beam of Figure 9a is a special case where the shear *V* and bending moment *M* can be represented by a single analytical function, that is

$$V(x) = w(L - x) \qquad (15a)$$

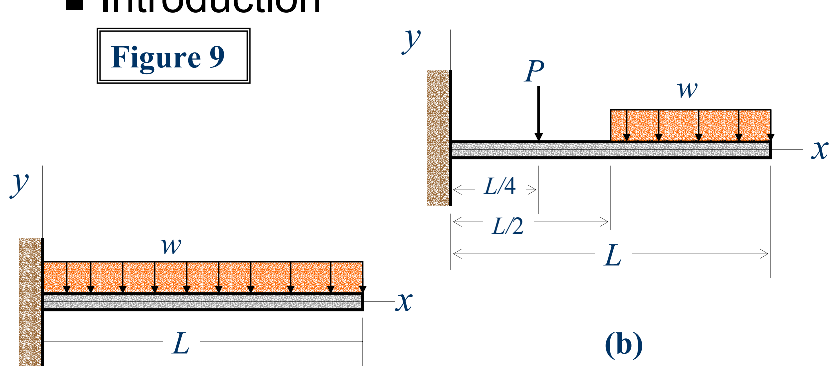
and

$$M(x) = w\left(-L^2 + 2Lx - x^2\right) \qquad (15b)$$

Singularity Functions

- Introduction

Figure 9



(a) (b)

Singularity Functions

- Introduction
  - While for the beam of Figure 9b, the shear *V* or moment *M* cannot be expressed in a single analytical function. In fact, they should be represented for the three intervals, namely

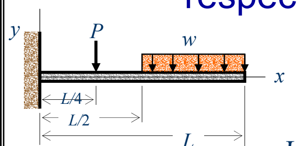
    $0 \leq x \leq L/4$,

    $L/4 \leq x \leq L/2$, and

    $L/2 \leq x \leq L$

Singularity Functions

- Introduction
  - For the three intervals, the shear *V* and the bending moment *M* can are given, respectively, by

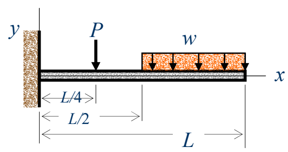


$$V(x) = \begin{cases} P + \dfrac{wL}{2} & \text{for } 0 \leq x \leq L/4 \\ \dfrac{wL}{2} & \text{for } L/4 \leq x \leq L/2 \\ \dfrac{wL}{2} - w\left(x - \dfrac{L}{2}\right) & \text{for } L/2 \leq x \leq L \end{cases}$$

Singularity Functions

- Introduction

  and



$$M(x)=\begin{cases} -\dfrac{PL}{4}-\dfrac{3wL^2}{8}+Px+\dfrac{wL}{2}x & \text{for } 0\le x\le L/4 \\ -\dfrac{3wL^2}{8}+\dfrac{wL}{2}x & \text{for } L/4\le x\le L/2 \\ -\dfrac{3wL^2}{8}+\dfrac{wL}{2}x-\dfrac{w}{2}\left(x-\dfrac{L}{2}\right)^2 & \text{for } L/2\le x\le L \end{cases}$$

Singularity Functions

- Introduction
  - We see that even with a cantilever beam subjected to two simple loads, the expressions for the shear and bending moment become complex and more involved.
  - *Singularity functions* can help reduce this labor by making *V* or *M* represented by a single analytical function for the entire length of the beam.

Singularity Functions

- Basis for Singularity Functions
  - Singularity functions are closely related to he unit step function used to analyze the transient response of electrical circuits.
  - They will be used herein for writing one bending moment equation (expression) that applies in all intervals along the beam, thus eliminating the need for matching equations, and reduce the work involved.

Singularity Functions

- Definition

  A singularity function is an expression for *x* written as $\langle x - x_0 \rangle^n$, where *n* is any integer (positive or negative) including zero, and x_0 is a constant equal to the value of *x* at the initial boundary of a specific interval along the beam.

Singularity Functions

- Properties of Singularity Functions
  - By definition, for $n \geq 0$,

$$\langle x - x_0 \rangle^n = \begin{cases} (x - x_0)^n & \text{when } x \geq x_0 \\ 0 & \text{when } x < x_0 \end{cases} \qquad (16)$$

  - Selected properties of singularity functions that are useful and required for beam-deflection problems are listed in the next slides for emphasis and ready reference.

Singularity Functions

- Selected Properties

$$\langle x - x_0 \rangle^n = \begin{cases} (x - x_0)^n & \text{when } n > 0 \text{ and } x \geq x_0 \\ 0 & \text{when } n > 0 \text{ and } x < x_0 \end{cases} \qquad (17)$$

$$\langle x - x_0 \rangle^0 = \begin{cases} 1 & \text{when } n > 0 \text{ and } x \geq x_0 \\ 0 & \text{when } n > 0 \text{ and } x < x_0 \end{cases} \qquad (18)$$

Singularity Functions

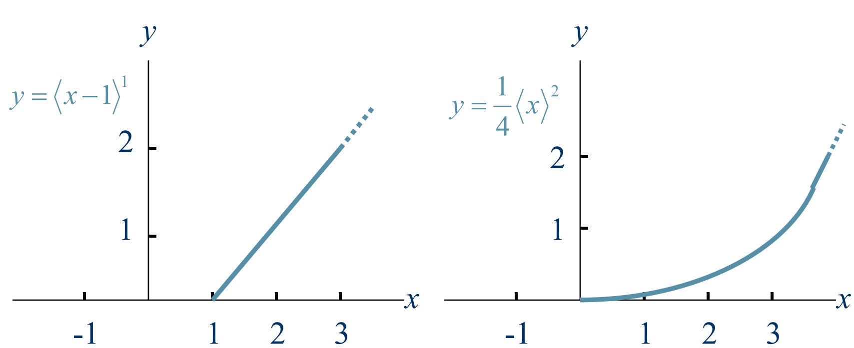
- Integration and Differentiation of Singularity Functions

$$\int \langle x - x_0 \rangle^n dx = \frac{1}{n+1} \langle x - x_0 \rangle^{n+1} + \mathrm{C} \quad \text{when } n > 0 \tag{19}$$

$$\frac{d}{dx} \langle x - x_0 \rangle^n = n \langle x - x_0 \rangle^{n-1} \quad \text{when } n > 0 \tag{20}$$

Singularity Functions

- Examples: Singularity Functions



(a) **Figure 10** (b)

Singularity Functions

- Examples: Singularity Functions

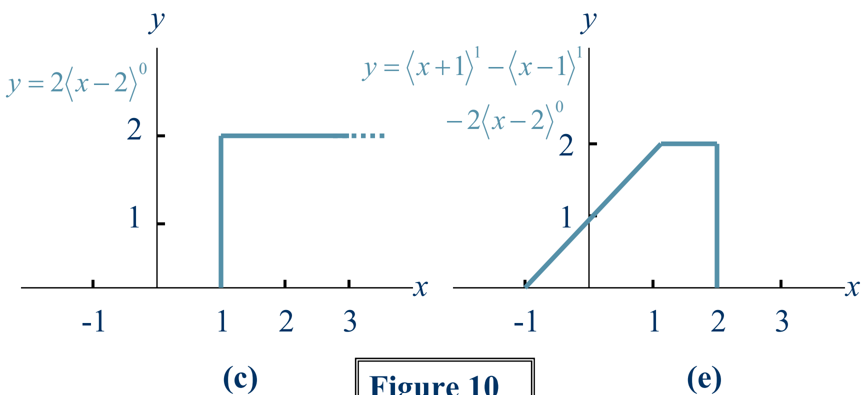


Figure 10

Singularity Functions

- Application of Singularity Functions in Developing a Single Equation to Describe the Bending Moment
  - By making use of singularity functions properties, a single equation (expression) for the bending moment for a beam can be obtained.
  - Also, the corresponding value of *M* in any interval can be computed.

Singularity Functions

- Application of Singularity Functions in Developing a Single Equation to Describe the Bending Moment
  – To illustrate this, consider the beam of the following figure (Fig.11).
  – The moment equations at the four designated sections are written as shown in the following slide.

Singularity Functions

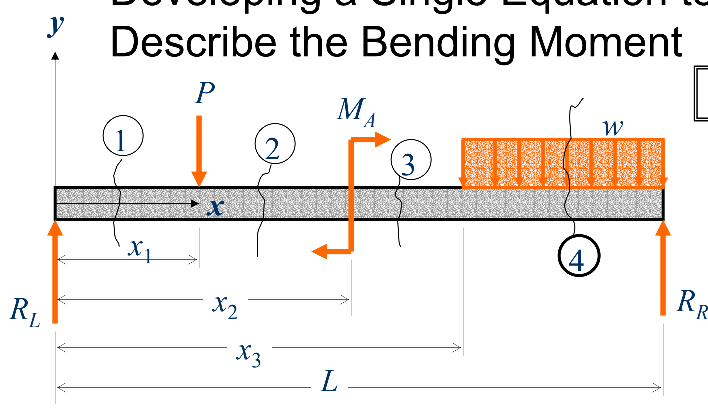
- Applications of Singularity Functions in Developing a Single Equation to Describe the Bending Moment

$$
\begin{aligned}
&M_1 = R_L x && \text{for } 0 < x < x_1 \\
&M_2 = R_L x - P(x - x_1) && \text{for } x_1 < x < x_2 \\
&M_3 = R_L x - P(x - x_1) + M_A && \text{for } x_2 < x < x_3 \qquad (21) \\
&M_4 = R_L x - P(x - x_1) + M_A - \frac{w(x - x_3)}{2} && \text{for } x_2 < x < x_L
\end{aligned}
$$

Singularity Functions

- Application of Singularity Functions in Developing a Single Equation to Describe the Bending Moment

Figure 11



Singularity Functions

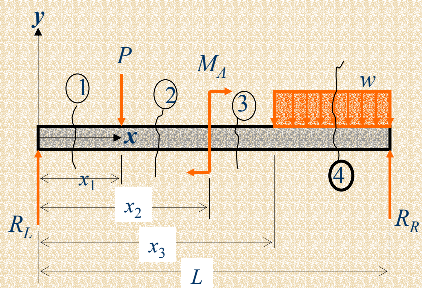
- Application of Singularity Functions in Developing a Single Equation to Describe the Bending Moment
  - These for moment equations can be combined into a single equations by means of singularity functions to give

$$M(x) = R_L x - P\langle x - x_1\rangle^1 + M_A\langle x - x_2\rangle^0 - \frac{w}{2}\langle x - x_3\rangle^2 \quad \text{for } 0 < x < L \qquad (22)$$

Where $M(x)$ indicates that the moment is a function of x.

Singularity Functions

- Typical Singularity Functions



$$M(x) = R_L x - P\langle x - x_1\rangle^1 + M_A\langle x - x_2\rangle^0 - \frac{w}{2}\langle x - x_3\rangle^2 \qquad \text{for } 0 < x < L \tag{22}$$

Singularity Functions

- Notes on Distributed Loads
  - When using singularity functions to describe bending moment along the beam length, special considerations must be taken when representing distributed loads, such as those shown in Figure 12.
  - The distributed load cannot be represented by a single function of *x* for all values of *x*.

Singularity Functions

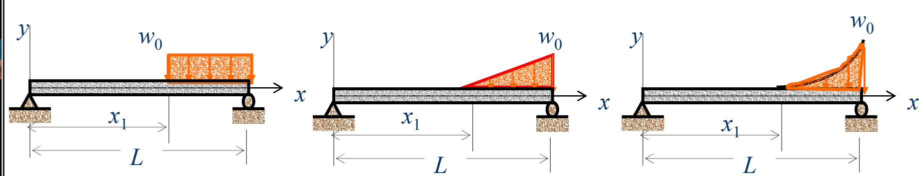
- Notes on Distributed Loads
  - The distributed loading should be an *open-ended to the right* (Figure 11) of the beam when we apply the singularity function.
  - A distributed loading which does not extend to the right end of the beam or which is discontinuous should be replaced as shown in Figures by an equivalent combination of open-ended loadings.

Singularity Functions

- Notes on Distributed Loads
  - Examples of open-ended-to-right distributed loads, which are ready for singularity function use are shown in Figure 12.
  - Examples of non open-ended-to-right or discontinuous distributed loads are shown in Figures 13, 14, and 15.

Singularity Functions

- Moment due to Distributed Loads



$$M_{w_0} = -\frac{w_0}{2}\langle x - x_1\rangle^2 \qquad M_{w_0} = -\frac{w_0}{6(L - x_1)}\langle x - x_1\rangle^3 \qquad M_{w_0} = -k\langle x - x_1\rangle^{n+2}$$

Figure 12. Open-ended-to-right distributed loads

Singularity Functions

- Moment due to Distributed Loads

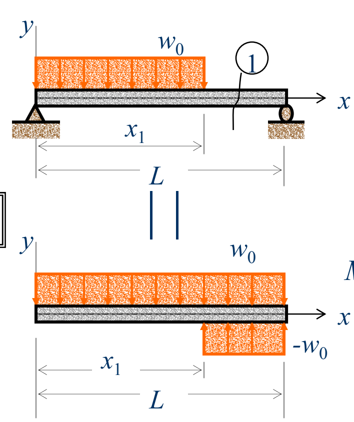


Figure 13

The moment at section 1 due to distributed load alone is

$$M_{w_0} = -\frac{w_0}{2}\langle x - 0\rangle^2 + \frac{w_0}{2}\langle x - x_1\rangle^2$$

Singularity Functions

- Moment due to Distributed Loads

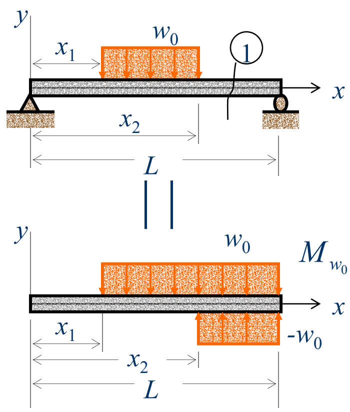


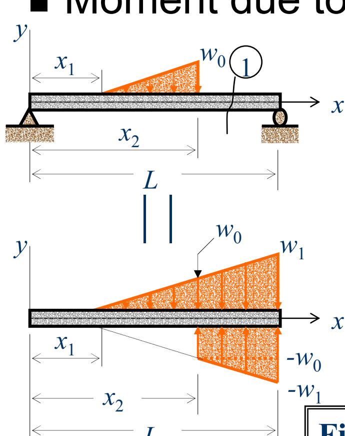
Figure 14

The moment at section 1 due to distributed load alone is

$$M_{w_0} = -\frac{w_0}{2}\langle x - x_1 \rangle^2 + \frac{w_0}{2}\langle x - x_2 \rangle^2$$

Singularity Functions

- Moment due to Distributed Loads



$$\frac{w_1}{w_0} = \frac{L - x_1}{x_2 - x_1} \Rightarrow w_1 = \frac{w_0 (L - x_1)}{x_2 - x_1}$$

The moment at section 1 due to distributed load alone is

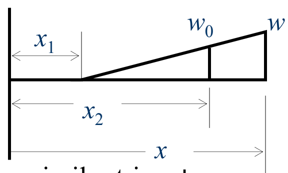
$$M_{w_0} = -\frac{w_0}{6(x_2 - x_1)}\langle x - x_1 \rangle^3 + \frac{w_0}{6(x_2 - x_1)}\langle x - x_2 \rangle^3 + \frac{w_0 \langle x - x_2 \rangle^2}{2}$$

Figure 15

Singularity Functions

- Moment due to Distributed Loads

  Note that in Fig. 14, the linearly varying load at any point $x \geq x_1$ is



$$w = \frac{w_0(x - x_1)}{x_2 - x_1}$$

From similar triangles:

$$\frac{w}{w_0} = \frac{x - x_1}{x_2 - x_1}$$

The moment of this load for Any point $x \geq x_1$ is

$$M = -\frac{1}{2}\left[\frac{w_0(x - x_1)}{x_2 - x_1}(x - x_1)\right]\left(\frac{x - x_1}{3}\right) = \frac{w_0}{6(x_2 - x_1)}(x - x_1)^3$$

Singularity Functions

- Example 4

  Use singularity functions to write the moment equation for the beam shown in Figure 16. Employ this equation to obtain the elastic curve, and find the deflection at x = 10 ft.

$E = 29 \times 10^6$ psi

$I = 464$ in^2

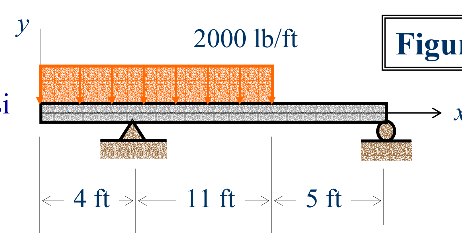
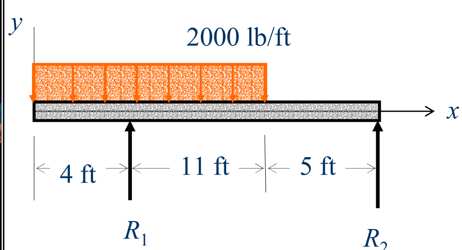


Figure 16

Singularity Functions

- Example 4 (cont'd)



Find the reactions:

$$+\circlearrowleft \sum M_2 = 0;\ R_1(16) - 2000(15)(5 + \frac{15}{2}) = 0$$

$$\therefore R_1 = 23{,}437.5 \text{ lb}$$

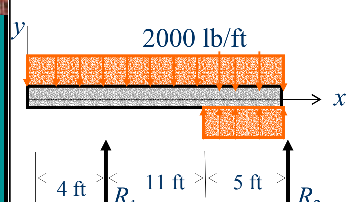
$$+\uparrow \sum F_y = 0;\ R_1 + R_2 - 2000(15) = 30{,}000$$

$$\therefore R_2 = 6{,}562.5 \text{ lb}$$

Singularity Functions

- Example 4 (cont'd)

  A single expression for the bending moment can be obtained using the singularity functions:



$$\boxed{M(x) = -\frac{2000(x)^2}{2} + \frac{2000\langle x-15\rangle^2}{2} + 23{,}437.5\langle x-4\rangle^1} \quad (23a)$$

Singularity Functions

- Example 4 (cont'd)

  The elastic curve is found by integrating Eq. 23 twice

$$EIy'' = M(x) = -\frac{2000x^2}{2} + \frac{2000\langle x-15\rangle^2}{2} + 23{,}437.5\langle x-4\rangle^1$$

$$EIy' = EI\theta = -2000\frac{x^3}{6} + \frac{2000\langle x-15\rangle^3}{6} + \frac{23{,}437.5\langle x-4\rangle^2}{2} + C_1 \quad (23b)$$

$$EIy = -2000\frac{x^4}{24} + \frac{2000\langle x-15\rangle^4}{24} + \frac{23{,}437.5\langle x-4\rangle^3}{6} + C_1x + C_2 \quad (23c)$$

Singularity Functions

- Example 4 (cont'd)

  Boundary conditions:

  $y = 0$ at $x = 4$ ft and at $x = 20$ ft

$$EIy(4) = 0 = -2000\frac{(4)^4}{24} + \underbrace{\frac{2000\langle 4-15\rangle^4}{24}}_{0} + \underbrace{\frac{23{,}437.5\langle 4-4\rangle^3}{6}}_{0} + C_1(4) + C_2$$

$$\therefore 4C_1 + C_2 = 21{,}333.3 \quad (23d)$$

$$EIy(20) = 0 = -2000\frac{(20)^4}{24} + \frac{2000\langle 20-15\rangle^4}{24} + \frac{23{,}437.5\langle 20-4\rangle^3}{6} + C_1(20) + C_2$$

$$\therefore 20C_1 + C_2 = -2{,}718{,}750 \quad (23e)$$

Singularity Functions

- Example 4 (cont'd)

  From Eqs 23d and 23e, the constants of integrations are found to

$$C_1 = 168{,}588.54 \quad \text{and} \quad C_2 = -653{,}020.88$$

Therefore, the elastic curve is given by

$$y(x) = \frac{1}{EI}\left[-2000\frac{(x)^4}{24} + \frac{2000\langle x-15\rangle^4}{24} + \frac{23{,}437.5\langle x-4\rangle^3}{6} + 168{,}588.5x - 653{,}020.9\right] \tag{23f}$$

Singularity Functions

- Example 4 (cont'd)

  The deflection at any point along the beam can be calculated using Eq. 23f (elastic curve equation). Therefore, the deflection *y* at *x* = 10 ft is

$$y(10) = \frac{1}{EI}\left[-2000\frac{(10)^4}{24} + \underbrace{\cancel{\frac{2000\langle 10-15\rangle^4}{24}}}_{0} + \frac{23{,}437.5\langle 10-4\rangle^3}{6} + 168{,}588.5(10) - 653{,}020.9\right]$$

$$y(10) = \frac{(12)^2}{29\times10^6(464)}[-833{,}333.3 + 843{,}750 + 1{,}685{,}885 - 653{,}020.9]$$

$$y_{10''} = 0.011165 \text{ ft} = \boxed{0.1339 \text{ in}}$$

Singularity Functions

- Example 5

  A beam is loaded and supported as shown in Figure 17. Use singularity functions to determine, in terms of *M*, *L*, *E*, and *I*,

  a) The deflection at the middle of the span.

  b) The maximum deflection of the beam.

Singularity Functions

- Example 5 (cont'd)

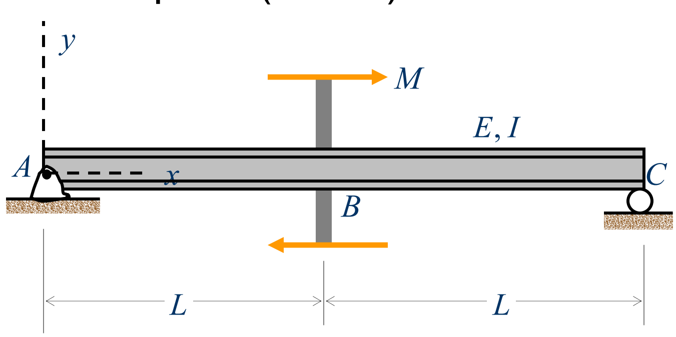
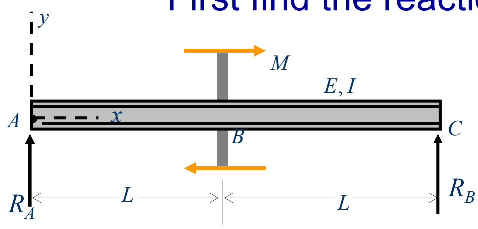


Figure 17

Singularity Functions

- Example 5 (cont'd)

First find the reactions R_A and R_B:



$$+\circlearrowright \sum M_C = 0;\ R_A(2L) + M = 0$$

$$\therefore R_A = -\frac{M}{2L}$$

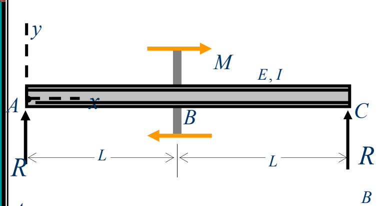
$$+\uparrow \sum F_y = 0;\ R_A + R_B = 0$$

$$\therefore R_B = \frac{M}{2L}$$

Singularity Functions

- Example 5 (cont'd)

Singularity functions to describe the bending moment:



$$M(x) = -\frac{Mx}{2L} + M\langle x - L\rangle^0 \quad \text{(24a)}$$

or

$$EIy'' = -\frac{Mx}{2L} + M\langle x - L\rangle^0 \quad \text{(24b)}$$

Singularity Functions

- Example 5 (cont'd)

Integrating Eq. 24.b twice, we get

$$EIy'' = -\frac{Mx}{2L} + M\langle x-L\rangle^0$$

$$EIy' = -\frac{Mx^2}{4L} + M\langle x-L\rangle^1 + C_1 \qquad (24c)$$

$$EIy = -\frac{Mx^3}{12L} + \frac{M\langle x-L\rangle^2}{2} + C_1x + C_2 \qquad (24d)$$

Singularity Functions

- Example 5 (cont'd)

Boundary conditions:

At $x = 0$, $y = 0$ Therefore: $C_2 = 0$

At $x = 2L$, $y = 0$ Therefore: $C_1 = ML/12$

Thus,

$$y = \frac{M}{48EIL}\left[-x^3 + 6L\langle x-L\rangle^2 + L^2x\right] \qquad (24e)$$

$$y(L) = \frac{M}{48EIL}\left[-L^3 + L^3\right] = \boxed{0} \qquad (24f)$$

Singularity Functions

- Example 5 (cont'd)

  (b) Finding the maximum deflection:

$$y' = \frac{M}{12EIL}\left[-3x^2 + 12L\langle x-L\rangle^1 + L^2\right] \qquad y_{\max} \text{ when } y' = 0$$

$$\text{Therefore: } -3x^3 + 12\langle x-L\rangle^1 + L^2 = 0 \;\Rightarrow\; x = L/\sqrt{3}$$

$$y_{\max} = y\left(L/\sqrt{3}\right) = \frac{M}{12EIL}\left[-\frac{L^3}{3\sqrt{3}} + \frac{L^3}{\sqrt{3}}\right] = \frac{2ML^2}{36\sqrt{3}EI}$$

$$= \boxed{\frac{\sqrt{3}ML^2}{54EI}}$$