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LECTURE

16

Chapter 9.1 – 9.3

BEAMS: DEFORMATION BY INTEGRATION

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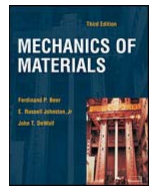

by

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ENES 220 – Mechanics of Materials

Department of Civil and Environmental Engineering

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Beam Deformation

- Introduction
  - In the previous chapter our concern with beams was to determine the flexural and transverse shear stresses is straight homogenous beams of uniform cross section.
  - We also dealt with various types of composite beams, and we're able to find these stresses with some method such as the transformed section.

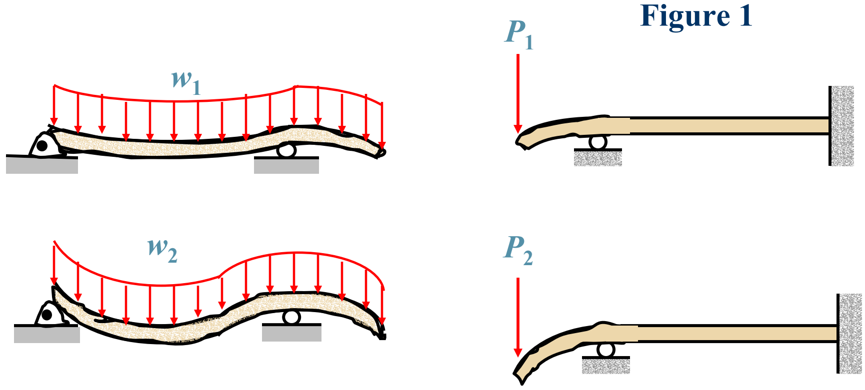
Beam Deformation

- Introduction
  – In this chapter our concern will be beam *deformation* (or deflection).
  – There are important relations between applied load and stress (flexural and shear) and the amount of deformation or deflection that a beam can exhibit.

Beam Deformation

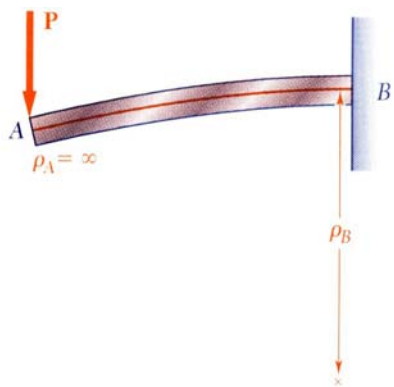
- Deflection of Beams

Figure 1



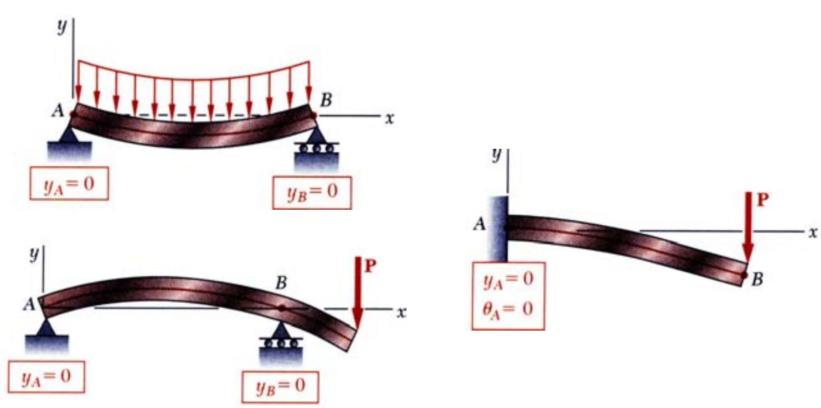
(a) $w_2 >> w_1$ (b) $P_2 >> P_1$

Beam Deformation



- Curvature varies linearly with x
- At the free end A, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$
- At the support B, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Beam Deformation



Beam Deformation

- Introduction
  - In design of beams, it is important sometimes to limit the deflection for specified load.
  - So, in these situations, it is not enough only to design for the strength (flexural normal and shearing stresses), but also for excessive deflections of beams.

Beam Deformation

- Introduction
  - Figure 1 shows generally two examples of how the amount of deflections increase with the applied loads.
  - Failure to control beam deflections within proper limits in building construction is frequently reflected by the development of cracks in plastered walls and ceilings.

Beam Deformation

- Introduction
  - Beams in many machines must deflect just right amount for gears or other parts to make proper contact.
  - In many instances the requirements for a beam involve:
    - **A given load-carrying capacity, and**
    - **A specified maximum deflection.**

Beam Deformation

- Methods for Determining Beam Deflections
  - The deflection of a beam depends on four general factors:
    1. Stiffness of the materials that the beam is made of,
    2. Dimensions of the beam,
    3. Applied loads, and
    4. Supports

Beam Deformation

- Methods for Determining Beam Deflections
  – Three methods are commonly used to find beam deflections:
    1) **The double integration method,**
    2) **The singularity function method, and**
    3) **The superposition method**

Beam Deformation

- General Load-Deflection Relationships
  – Whenever a real beam is loaded in any manner, the beam will deform in that an initially straight beam will assume some deformed shape such as those illustrated (with some exaggerations) in Figure 1
  – In a well-designed structural beam the deformation of the beam is usually undetectable to the naked eye.

Beam Deformation

- General Load-Deflection Relationships
  - On the other hand, the bending of a swimming pool diving board is quite observable.
  - The word *deflection* generally refers to the deformed shape of a member subjected to bending loads.
  - The deflection is used in reference to the deformed shape and position of the longitudinal *neutral axis* of a beam.

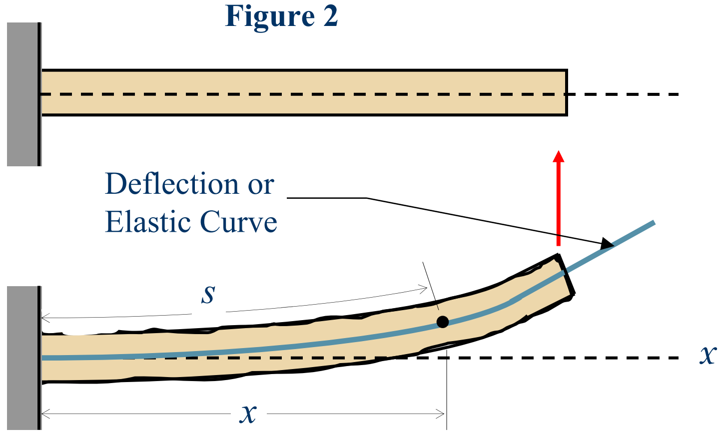
Beam Deformation

- General Load-Deflection Relationships
  - In the deformed condition the neutral axis, which his initially a straight longitudinal line, assumes some particular shape that is called the ***deflection curve***.
  - The derivation of this curve from its initial position at any point is called the deflection at that point.

Beam Deformation

- General Load-Deflection Relationships

Figure 2



Beam Deformation

- Elastic Curve
  - The Differential Equation
    - The differential equation that governs beam deflection will now be developed.
    - The basis for this differential equation, plus more other approximations, is that plane sections within the beam remain plane before and after loading, and the deformation of the fibers (elongation and contraction) is proportional to the distance from N.A.

Beam Deformation

- Development of the Differential Equation
  - Consider the beam shown in Fig. 3 that is subjected to the couple shown.

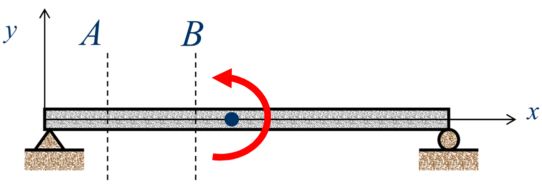


Figure 3

Beam Deformation

- Development of the Differential Equation

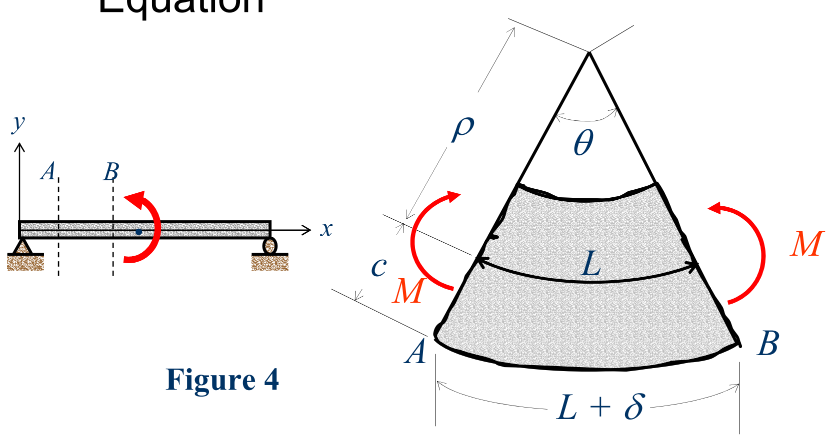


Figure 4

Beam Deformation

- Development of the Differential Equation
  - In region of constant bending moment, the elastic curve is an arc of a circle of radius ρ as shown in Fig. 4.
  - Since the portion *AB* of the beam is bent only with couples, sections *A* and *B* remain plane as indicated earlier.

Beam Deformation

- Development of the Differential Equation

  From Fig. 4, we have

$$\theta = \frac{L}{\rho} = \frac{L+\delta}{\rho + c} \qquad (1)$$

Or

$$\frac{\rho + c}{\rho} = \frac{L+\delta}{L} \qquad (2)$$

Beam Deformation

- Development of the Differential Equation
  - Dividing the right left-hand side of Eq. 2 by ρ, yields

$$\frac{1+\frac{c}{\rho}}{1}=\frac{L+\delta}{L}$$

or

$$\frac{c}{\rho}=\frac{L+\delta}{L}-1=\frac{L+\delta-L}{L}$$

Therefore,

$$\frac{c}{\rho}=\frac{\delta}{L} \qquad (3)$$

Beam Deformation

- Development of the Differential Equation

But $\dfrac{\delta}{L}=\text{strain }\varepsilon=\dfrac{\sigma}{E}$, and $\sigma=\dfrac{Mc}{I}$

Therefore,

$$\frac{c}{\rho}=\frac{Mc}{EI}$$

or

$$\frac{1}{\rho}=\frac{M}{EI} \qquad (4)$$

Beam Deformation

- Development of the Differential Equation
  - Eq. 4 for the elastic curvature of the elastic curve is useful only when the bending moment is constant for the interval of the beam involved.
  - For most beams, however, the moment is a function of position along the beam as was seen in Chapter 8.

Beam Deformation

- Development of the Differential Equation

  Recall from calculus, the curvature is given by

$$\frac{1}{\rho} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \qquad (5)$$

Beam Deformation

- Development of the Differential Equation
  - Eq. 5 is difficult to apply in real situation.
  - However, if we realize that for most beams the slope *dy/dx* is very small, and its square is much smaller, then term

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

$$\left(\frac{dy}{dx}\right)^2$$

in Eq. 5 can be neglected as compared to unity.

Beam Deformation

- Development of the Differential Equation

With this assumption on the slope *dy/dx* being very small quantity, Eq. 5 becomes

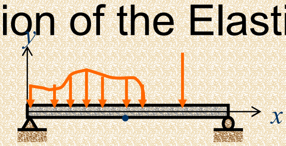
$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \qquad (6)$$

Combining Eqs. 4 and 6, we get

$$EI\frac{d^2y}{dx^2} = M(x) \qquad (7)$$

Beam Deformation

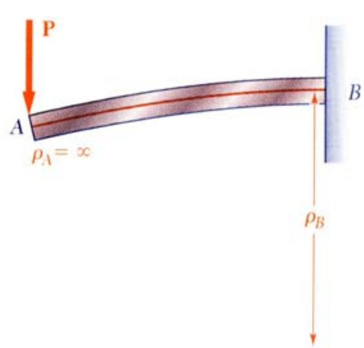
- The Differential Equation of the Elastic Curve for a Beam



$$EI\frac{d^2y}{dx^2} = M(x) \qquad (8)$$

E = modulus of elsticity for the material
I = moment of inertia about the neutral axis of cross section
$M(x)$ = bending moment along the beam as a function of x

Beam Deformation



- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

Beam Deformation

- From elementary calculus, simplified for beam parameters,

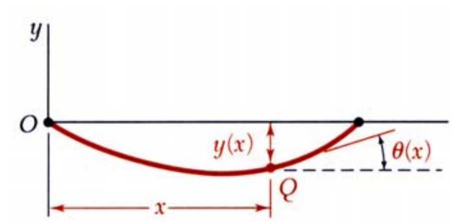
$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

- Substituting and integrating,

$$EI\frac{1}{\rho} = EI\frac{d^2y}{dx^2} = M(x)$$

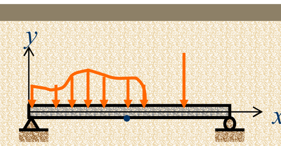
$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)dx + C_1x + C_2$$



Beam Deformation

- Sign Convention



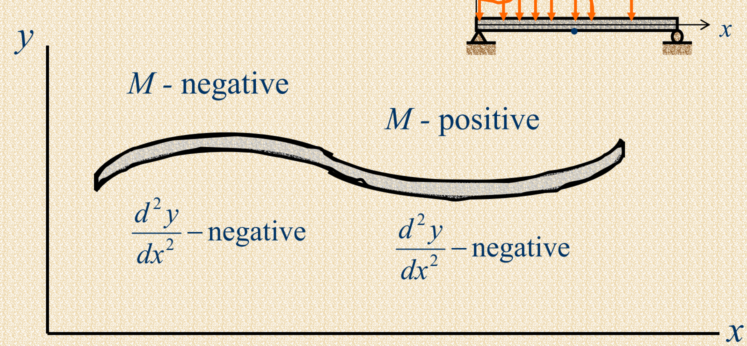
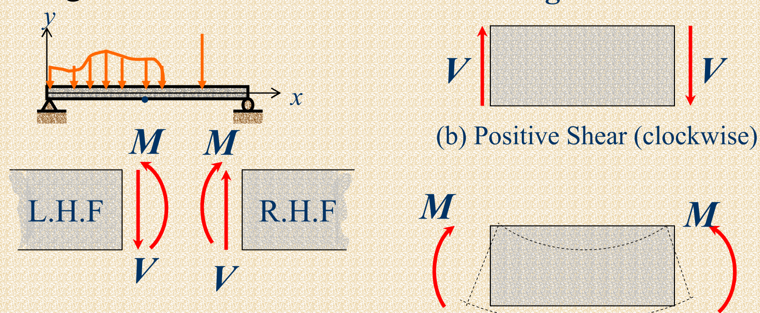


Figure 5. Elastic Curve

Beam Deformation

- Sign Convention

Figure 6



(a) Positive Shear & Moment

(b) Positive Shear (clockwise)

(c) Positive Moment (concave upward)

Beam Deformation

- Relation of the Deflection *y* with Physical Quantities such as *V* and *M*

$$
\begin{aligned}
\text{deflection} &= y \\
\text{slope} &= \frac{dy}{dx} \\
\text{moment}\,(M) &= EI\frac{d^2y}{dx^2} \\
\text{shear}\,(V) &= \frac{dM}{dx} = EI\frac{d^3y}{dx^3}\ (\text{for } EI \text{ constant}) \\
\text{load}\,(w) &= \frac{dV}{dx} = EI\frac{d^4y}{dx^4}\ (\text{for } EI \text{ constant})
\end{aligned}
\qquad (9)
$$

Beam Deformation

- Construction of Slope and Deflection Diagrams
  - In Chapter 8, a method based on the previous relations was presented for starting from the load diagram and drawing first the shear diagram and then the moment diagram.
  - This method can readily be extended to the construction of slope diagram and deflection diagram.

Beam Deformation

- Equation for Slope Diagram

  Note that from Eq. 9, the moment is given by

$$M = EI\frac{d^2y}{dx} = EI\frac{d}{dx}\left(\frac{dy}{dx}\right) = EI\frac{d\theta}{dx}$$

Slope θ

from which

$$\theta_{A\text{-}B} = \int_{\theta_A}^{\theta_B} d\theta = \int_{x_A}^{x_B} \frac{M}{EI}dx \qquad (10)$$

Beam Deformation

- Equation for Deflection Diagram

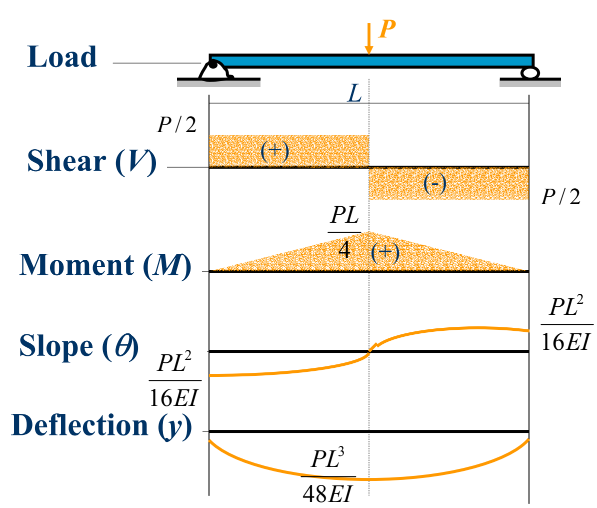
  Note that from Eq. 9, the slope is given by

$$\theta = \frac{dy}{dx}$$

from which

$$y_{A-B} = \int_{x_A}^{x_B} \theta dx = \int_{x_A}^{x_B}\int_{x_A}^{x_B} \frac{M}{EI} dx \qquad (11)$$

Beam Deformation



- Figure 7
  - Complete Series of Diagrams for Simply Supported beam

Beam Deformation

- Assumptions on Elastic Curve Equation
  1. The square of the slope of the beam is negligible compared to unity.
  2. The beam deflection due to shearing stresses is negligible (plane section remains plane)
  3. The values of *E* and *I* remain constant along the beam. If they are constant, and can be expressed as functions of *x*, then a solution using Eq. 8 is possible.

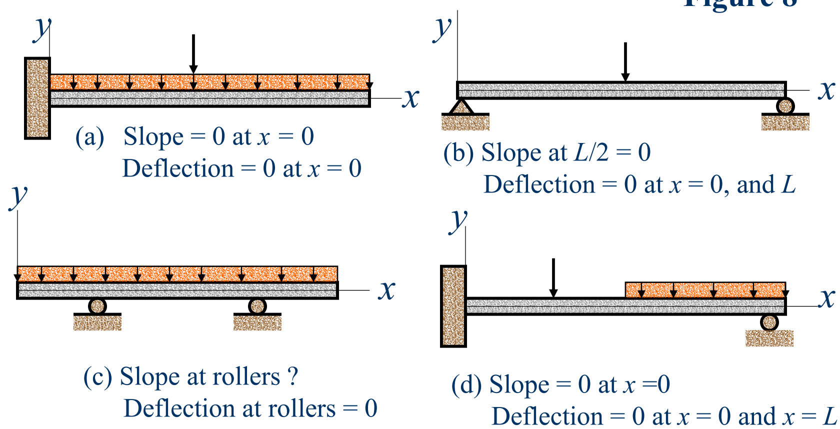
Deflection by Integration

- Boundary Conditions
  - Definition:
    - A boundary condition is defined as a known value for the deflection y or the slope θ at a specified location along the length of the beam. One boundary condition can be used to determine one and only one constant of integration.

Deflection by Integration

- Example Boundary Conditions

Figure 8



Deflection by Integration

- General Procedure for Computing Deflection by Integration
  1. Select the interval or intervals of the beam to be used; next, place a set of coordinate axes on the beam with the origin at one end of an interval and then indicate the range of values of *x* in each interval.
  2. List the variable boundary and matching

Deflection by Integration

- General Procedure for Computing Deflection by Integration (cont'd)

   Conditions for each interval selected.

   3. Express the bending moment M as a function of x for each interval selected and equate it to $EI(d^2y/dx^2)$.
   4. Solve the differential equation from step 3 and evaluate all constants of integration. Calculate y at specific points when required

Deflection by Integration

- Example 1

  A beam is loaded and supported as shown in the figure.

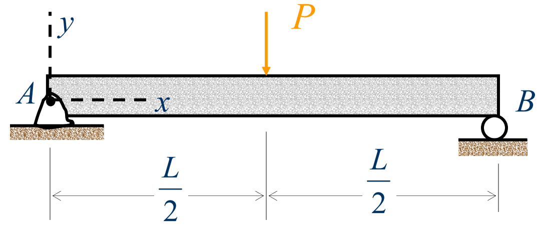
  a) Derive the equation of the elastic curve in terms of P, L, x, E, and I.

  b) Determine the slope at the left end of the beam.

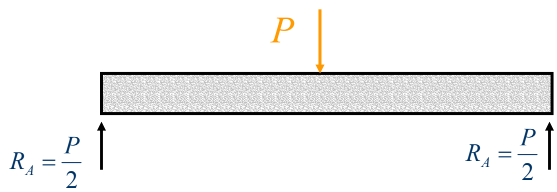
  c) Determine the deflection at $x = L/2$.

Deflection by Integration

- Example 1 (cont'd)

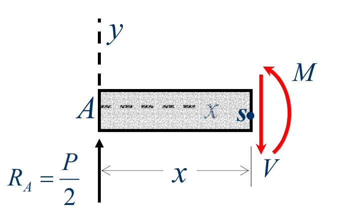


FBD



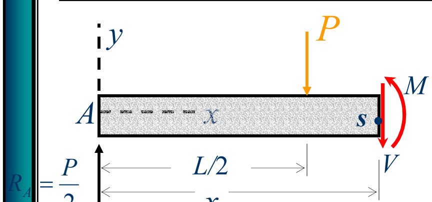
Deflection by Integration

- Example 1 (cont'd)



$$+\curvearrowleft \sum M_S = 0; -M + \frac{P}{2}x = 0$$

$$\Rightarrow \boxed{M = \frac{P}{2}x \quad \text{for } 0 \le x \le L/2} \quad (12a)$$



$$\boxed{M = \frac{P}{2}x - P\left(x - \frac{L}{2}\right) \quad \text{for } L/2 \le x \le L}$$

(12b)

Deflection by Integration

- Example 1 (cont'd)

Boundary conditions:

- θ = 0 at *x* = *L*/2 (from symmetry)
- *y* = 0 at *x* = 0 and *x* = *L*

Using Eqs 8 and 12a:

$$EI\frac{d^2y}{dx^2} = M(x) = \frac{P}{2}x$$

$$EI\frac{dy}{dx} = EI\theta = \int M(x)dx = \int \frac{P}{2}x$$

$$EI\theta = \frac{P}{2}\frac{x^2}{2} + C_1 = \frac{P}{4}x^2 + C_1 \qquad (12c)$$

Deflection by Integration

- Example 1 (cont'd)

Expression for the deflection *y* can be found by integrating Eq. 12c:

$$EIy = \int EI\theta dx = \int\left(\frac{P}{4}x^2 + C_1\right)$$

$$EIy = \frac{P}{4}\frac{x^3}{3} + C_1x + C_2$$

$$EIy = \frac{P}{12}x^3 + C_1x + C_2 \qquad (12d)$$

Deflection by Integration

- Example 1 (cont'd)

The objective now is to evaluate the constants of integrations C_1 and C_2:

$$EI\theta(L/2)=0$$

$$\frac{P}{4}x^2+C_1=0\Rightarrow\frac{P}{4}\left(\frac{L^2}{4}\right)+C_1=0$$

$$\boxed{C_1=-\frac{PL^2}{16}} \qquad (12e)$$

Deflection by Integration

- Example 1 (cont'd)

$$y(0)=0$$

$$\frac{P}{12}x^3+C_1x+C_2=0\Rightarrow\frac{P}{12}(0)+C_1(0)+C_2=0$$

$$\boxed{C_2=0} \qquad (12f)$$

(a)The equation of elastic curve from Eq. 12d

$$EIy=\frac{P}{12}x^3+C_1x+\cancelto{0}{C_2}$$

$$\boxed{y=\frac{1}{EI}\left(\frac{P}{12}x^3-\frac{PL^2}{16}x\right)} \qquad (12g)$$

Deflection by Integration

- Example 1 (cont'd)
  - (b) Slope at the left end of the beam:
    - From Eq. 12c, the slope is given by

$$EI\theta = \frac{P}{4}x^2 + C_1 = \frac{P}{4}x^2 - \frac{PL^2}{16}$$

$$\theta = \frac{1}{EI}\left(\frac{P}{4}x^2 - \frac{PL^2}{16}\right)$$

    - Therefore,

$$\theta_A = \theta(0) = \frac{1}{EI}\left(\frac{P}{4}(0)^2 - \frac{PL^2}{16}\right) = \boxed{-\frac{PL^2}{16}}$$

Deflection by Integration

- Example 1 (cont'd)
  – (c) Deflection at $x = L/2$:
    - From Eq. 12g of the elastic curve:

$$y = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{PL^2}{16}x\right)$$

$$y(L/2) = \frac{1}{EI}\left(\frac{P}{12}\left(\frac{L}{2}\right)^3 - \frac{PL^2}{16}\left(\frac{L}{2}\right)\right)$$

$$= \boxed{-\frac{PL^3}{48EI}}$$

Deflection by Integration

- Example 2

  A beam is loaded and supported as shown in the figure.

  a) Derive the equation for the elastic curve in terms of *w*, *L*, *x*, *E*, and *I*.

  b) Determine the slope at the right end of the beam.

  c) Find the deflection at *x* = *L*.

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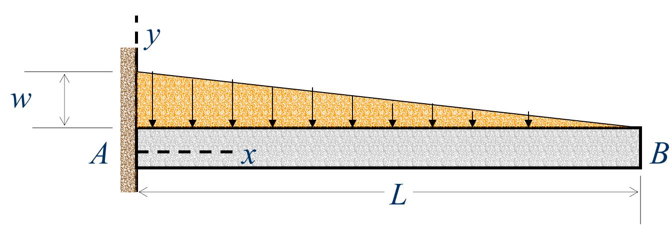
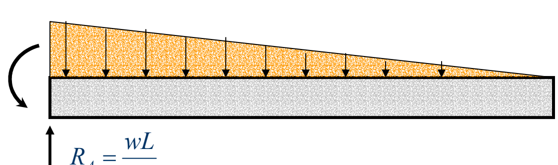
Deflection by Integration

- Example 2 (cont'd)



FBD

$$M_A = \frac{wL}{2}\left(\frac{2L}{3}\right) = \frac{wL^2}{6}$$

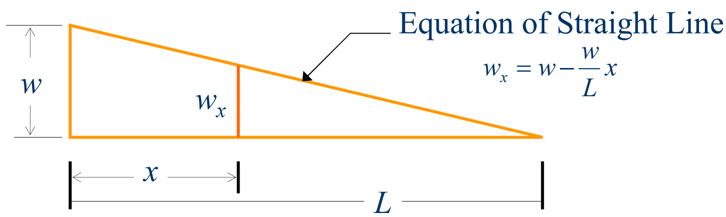


$$R_A = \frac{wL}{2}$$

Deflection by Integration

- Example 2 (cont'd)

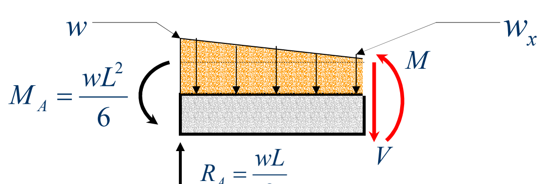
Find an expression for a segment of the distributed load:



$$\frac{w_x}{L-x} = \frac{w}{L} \quad \Rightarrow \quad \boxed{w_x = \frac{w(L-x)}{L} = w - \frac{w}{L}x} \qquad (13a)$$

Deflection by Integration

- Example 2 (cont'd)



$$+\circlearrowleft \sum M_s = 0; -M - \frac{wL^2}{6} + \frac{wL}{2}x - (w_x x)\frac{x}{2} - \frac{(w - w_x)x}{2}\frac{2x}{3} = 0$$

or

$$M(x) = -\frac{wL^2}{6} + \frac{wL}{2}x - \frac{w_x}{2}x^2 - \frac{(w - w_x)x^2}{3} \qquad (13b)$$

Deflection by Integration

- Example 2 (cont'd)
  - The solution for parts (a), (b), and (c) can be completed by substituting for w_x into Eq. 13b, equating the expression for $M(x)$ to the term $EI(d^2y/dx^2)$, and integrating twice to get the elastic curve and expression for the slope.
  - Note that the boundary conditions are that both the slope and deflection are zero at $x = 0$.

$$\text{i.e.;}\ \ EIy'' = EI\frac{d^2y}{dx^2} = M(x)$$

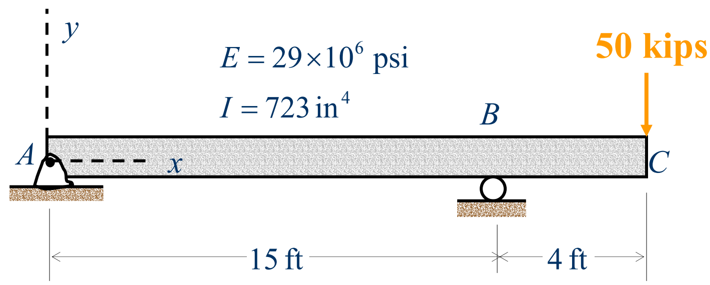
Deflection by Integration

- Example 3

For the overhanging steel beam *ABC* that subjected to concentrated load of 50 kips as shown, (a) derive an expression for the elastic curve, (b) determine the maximum deflection, and (c) find the slope at point *A*. The modulus of elasticity E is 29×10^6 psi and the moment of inertia of the cross section of the beam was found as 723 in^4.

Deflection by Integration

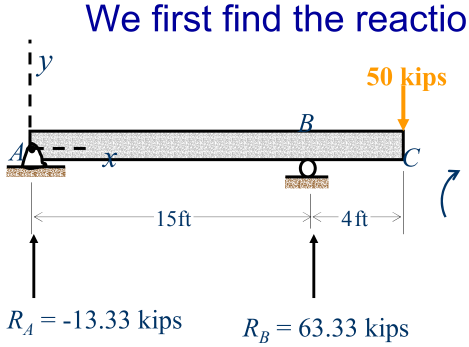
- Example 3 (cont'd)



Deflection by Integration

- Example 3 (cont'd)

We first find the reactions as follows:



$$+\sum M_B = 0;\ R_A(15) + 50(4) = 0$$

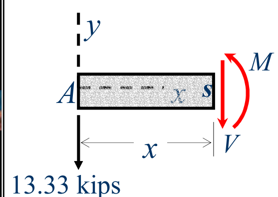
$$\therefore R_A = -13.33 \text{ kips}$$

$$+\uparrow \sum F_y = 0;\ R_A + R_B - 50 = 0$$

$$\therefore R_B = 50 - R_A = 50 - (-13.33) = 63.33 \text{ kips}$$

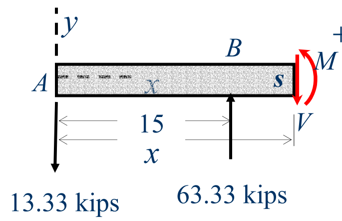
Deflection by Integration

- Example 3 (cont'd)



$$+\curvearrowleft \sum M_S = 0; -M - 10.53x = 0$$

$$\boxed{\therefore M = -13.33x} \qquad (14a)$$



$$+\curvearrowleft \sum M_S = 0; -M - 13.33x + 63.33(x - 15) = 0$$

$$\therefore M = -13.33x + 63.33x - 949.95$$

$$\boxed{M = -50x - 949.95} \qquad (14b)$$

Deflection by Integration

- Example 3 (cont'd)

Boundary conditions:

y(0) = 0, and *y*(15)=0

$$EIy'' = M(x) = -13.33x \quad \text{for } 0 \le x \le 15$$

$$EIy' = EI\theta = -13.33\frac{x^2}{2} + C_1 = -6.67x^2 + C_1 \qquad (14c)$$

$$EIy = -6.67\frac{x^3}{3} + C_1 x + C_2 \qquad (14d)$$

$$EIy(0) = 0 = -6.67\frac{(0)^3}{3} + C_1(0) + C_2$$

$$\therefore \mathrm{C}_2 = 0 \qquad (14e)$$

Deflection by Integration

- Example 3 (cont'd)

$$EIy(15) = 0 = -6.67\frac{(15)^3}{3} + C_1(15) + 0$$

$$\therefore C_1 = 500.25 \qquad (14f)$$

(a) The elastic curve is

$$\boxed{y(x) = \frac{1}{EI}\left(-6.67\frac{x^3}{3} + 500.25x\right) \text{ ft}} \qquad (14g)$$

Deflection by Integration

- Example 3 (cont'd)

(b) Maximum deflection occurs when the slope is zero. So setting *dy/dx* = θ = 0 in Eq. 14c, gives

$$y' = \theta = 0 = \frac{1}{EI}\left(-6.67x^2 + 500.25\right)$$

$$\therefore x = 8.66 \text{ ft}$$

Using Eq. 14g with x_{max} = 8.66 ft, gives

Conversion factor

$$y_{max} = \frac{(12)^2}{29\times10^3(723)}\left(-6.67\frac{(8.66)^3}{3} + 500.25(8.66)\right) = 0.01984 \text{ ft} = \boxed{0.238 \text{ in}}$$

Deflection by Integration

- Example 3 (cont'd)
  - (c) The slope at point *A* (*x* = 0) can be computed from Eq. 14c by substituting zero for *x* as follows:

$$y'(0) = \theta(0) = \frac{1}{EI}\left(-6.67x^2 + 500.25\right)$$

$$\therefore \theta_A = \frac{(12)^2}{29\times10^3}\left[-6.67(0)^2 + 500.25\right] = \boxed{0.00344 \text{ rad}}$$