CHAPTER

4c

Risk Analysis in Engineering and Economics

RELIABILITY ASSESSMENT

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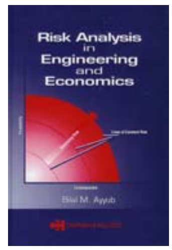

Risk Analysis for Engineering

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CHAPMAN
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Bayesian Methods

- Estimating Binomial Distribution
  - The function $p(t)$ is the time to failure cumulative distribution function, whereas $(1 - p(t))$ is the reliability or survivor function.
  - An estimate of the failure probability, p, is

$$\hat{p} = \frac{r}{n}$$

which is also the maximum likelihood estimate

Bayesian Methods

- Estimating Binomial Distribution (cont'd)
  - In order to obtain the Bayesian estimate for the probability p, a binomial test, in which the number of units n placed tested is fixed in advance, is considered.
  - The probability distribution of the number, r, of failed units during the test is given by the binomial distribution probability density function with parameters n and r as follows:

$$f(r; n, p) = \frac{n!}{(n-r)!\,r!} p^r (1-p)^{n-r} \qquad (73)$$

Bayesian Methods

- Estimating Binomial Distribution (cont'd)

Where

f = binomial probability mass function

r = random variable,

n and p = binomial distribution parameters.

  - The corresponding likelihood function is given by

$$l(p \mid r) = c\, p^r (1-p)^{n-r} \qquad (74)$$

where c is a constant which does not depend on the parameter of interest, p, and can be assigned a value of one since the constant c drops out from the posterior prediction equation.

Bayesian Methods

- Estimating Binomial Distribution (cont'd)
  – For any continuous prior distribution of parameter p with probability density function $h(p)$, the corresponding posterior probability density function can be written as

$$f(p \mid r) = \frac{p^r (1 - p)^{n-r} h(p)}{\int_{-\infty}^{\infty} p^r (1 - p)^{n-r} h(p) dp} \qquad (75)$$

Bayesian Methods

- Estimating Binomial Distribution (cont'd)
  – In order to better understand the difference between statistical inference and Bayes' estimation, the following case of the uniform prior distribution is discussed.
  – The prior distribution in this case is the standard uniform distribution, which is given by:

$$h(p) = \begin{cases} 1 & 0 \le p \le 1 \\ 0 & otherwise \end{cases} \qquad (76)$$

Bayesian Methods

- Estimating Binomial Distribution (cont'd)
  - Based on Eq. 75, the respective posterior distribution can be written as

$$f(p|r) = \frac{p^{(r+1)-1}(1-p)^{(n-r+1)-1}}{\int_0^1 p^{(r+1)-1}(1-p)^{(n-r+1)-1}\,dp} \qquad (77)$$

  - The posterior probability density function of 77 is the probability density function of the beta distribution.

Bayesian Methods

- Estimating Binomial Distribution (cont'd)
  - The mean value of this distribution, which is the Bayes' estimate of interest $p_{posterior}$ is given by

$$p_{posterior} = \frac{r+1}{n+2} \qquad (78)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution
  - A sample of n failure times from the exponential distribution, among which only r are distinct times to failure $t_1 < t_2 < \ldots < t_r$, and $n - r$ times to censoring $t_{c1}, t_{c2}, \ldots, t_{c(n-r)}$, so that the so-called total time on test, T, is given by

$$T = \sum_{i=1}^{r} t_i + \sum_{i=1}^{n-r} t_{ci} \qquad (88)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - Using the gamma distribution as the prior distribution of parameter λ, it is convenient to write the probability density of gamma distribution as a function of λ in the following form:

$$h(\lambda;\delta,\rho) = \frac{1}{\Gamma(\delta)} \rho^{\delta} \lambda^{\delta-1} e^{-\rho\lambda} \qquad (89)$$

where the parameters

$$\lambda > 0, \rho \geq 0, \text{ and } \delta \geq 0$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - These parameters can be interpreted as having δ fictitious failures in p total time leading to $\lambda = \delta/p$.
  - For the time being these parameters are assumed known.
  - Also, it is assumed that the quadratic loss function of Eq. 70 is used.

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - For the exponential time-to-failure data, the likelihood function can be written as

$$l(\lambda \mid t) = f(t_1)f(t_2)\cdots f(t_r)R(t_{c,1})R(t_{c,2})\cdots R(t_{c,n-r}) \qquad \text{(90a)}$$

Where

$f(t_i)$ = probability density function at time to failure t_i

$R(t_{c,i})$ = the reliability value at the time to censoring $t_{c,i}$.

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  – Therefore, the following likelihood function can be obtained:

$$l(\lambda \mid t) = \prod_{i=1}^{r} \lambda e^{-\lambda t_i} \prod_{j=1}^{n-r} e^{-\lambda t_{cj}} = \lambda^r e^{-\lambda T} \qquad (90b)$$

where T is the total time on test as given by Eq. 88.

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  – Using the Bayes' theorem with the prior distribution given by Eq. 89 and the likelihood function of Eq. 90, one can find the posterior density function of the parameter, λ, as:

$$f(\lambda \mid T) = \frac{e^{-\lambda(T+\rho)}\,\lambda^{r+\delta-1}}{\int_0^{\infty} \lambda^{r+\delta-1} e^{-\lambda(T+\rho)}\,d\lambda} \qquad (91)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - Recalling the definition of the gamma function of Eq. 80, the integral in the denominator of Eq. 91 is

$$f(\lambda \mid T) = \frac{(\rho + T)^{\delta + r}}{\Gamma(\delta + r)} \lambda^{r+\delta-1} e^{-\lambda(T+\rho)}$$

or

$$\int_0^\infty \lambda^{r+\delta-1} e^{-\lambda(T+\rho)} d\lambda = \frac{\Gamma(\delta + r)}{(\rho + T)^{\delta + r}}$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - Finally, the posterior probability density function of λ can be written as

$$f(\lambda \mid T) = \frac{(\rho + T)^{\delta + r}}{\Gamma(\delta + r)} \lambda^{r+\delta-1} e^{-\lambda(T+\rho)} \qquad (92)$$

  - Comparing the above function with the prior one of Eq. 89 reveals that the posterior distribution is also a gamma distribution with parameters

$$\rho' = r + \delta \quad \text{and} \quad \lambda' = T + \rho$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - Since a quadratic loss function is assumed, the point Bayesian estimate of λ is the mean of the posterior gamma distribution with parameters ρ' and λ'.
  - Therefore, the point Bayesian estimate, $\lambda_{posterior}$, can be obtained as

$$\lambda_{posterior} = \frac{\rho'}{\lambda'} = \frac{r + \delta}{T + \rho} \qquad (93)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - The corresponding probability intervals can be obtained using Eq. 72. For example, the 100(1 - α) level upper one-sided Bayes' probability interval for λ can be obtained from the following equation based on the posterior distribution Eq. 92:

$$Pr(\lambda < \lambda_u) = 1 - \alpha \qquad (94)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - The same upper one-sided probability interval for λ can be expressed in a more convenient form similar to the classical confidence interval, i.e., in terms of the chi-square distribution, as follows:

$$Pr\left(2\lambda(\rho+T) < \chi^2_{1-\alpha}\, 2(\delta+r)\right) = 1 - \alpha \qquad (95)$$

such that

$$\lambda_u = \frac{\chi^2_{1-\alpha,2(\delta+r)}}{2(\rho+T)} \qquad (96)$$

Bayesian Methods

- Parameter Estimation for the Exponential Distribution (cont'd)
  - Contrary to classical estimation, the number of degrees of freedom, $2(\delta + r)$, for the Bayes' probability limits is not necessarily integer.
  - The chi-square value in Eq. 96 can be obtained from tables of the chi-square probability distribution available in probability and statistics textbooks, such as Ayyub and McCuen (2003).

Bayesian Methods

Table 22. Relating the Coefficient of Variation to Prior Shape and Scale Parameters for the Gamma Distribution

Shape Parameter (δ) as a Prior Number of Failures	Scale Parameter (p) as a Prior Total Time on test	Coefficient of Variation (%)
1	100	100
5	500	45
10	1000	32
100	10000	10

Reliability Analysis of Systems

- The objective herein is to provide, develop, and demonstrate methods needed for assessing hazard functions of most widely used system models.
- Systems are assumed to be composed of components that have statistically independent failure events.
- The reliability functions for these components are defined based on the techniques discussed earlier.

Reliability Analysis of Systems

- System Failure Definition
  - Reliability block diagram (RBD) can be used to represent the structure of a system.
  - A reliability block diagram is a success-oriented network describing the function of the system.
  - For most systems considered below, the reliability functions can be evaluated based on their RBD.

Reliability Analysis of Systems

- System Failure Definition
  - Reliability assessment at the system starts with fundamental system modeling, i.e., series and parallel systems, and proceeds to more complex systems.
  - Additional information on functional modeling and system definition is provided in Chapter 3.

Reliability Analysis of Systems

- Series Systems
  - A series system composed of n components functions if and only if all of its n components are functioning.
  - Figure 16 shows an example of the RBD of a series system consisting of three components.

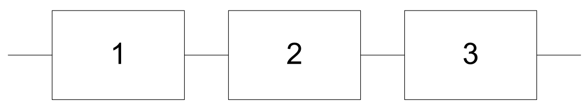

Figure 16. Series System Composed of Three Components

Reliability Analysis of Systems

- Series Systems (cont'd)
  - Reliability function of a series system composed of n components, $R_s(t)$, is given by

$$R_s(t) = \prod_{i=1}^{n} R_i(t) \qquad (97)$$

  - where $R_i(t)$ is the reliability function of ith component. If a series system is composed of identical components with reliability functions, $R_c(t)$, Eq. 97 is reduced to

$$R_s(t) = (R_c(t))^n \qquad (98)$$

Reliability Analysis of Systems

- Series Systems (cont'd)
  - Relationship between the system cumulative hazard rate function, $H_s(t)$, (CHRF) and the CHRFs of its components, $H_i(t)$, can be written as

$$H_s(t) = \sum_{i=1}^{n} H_i(t) \qquad (99a)$$

  - By taking derivative of $H_s(t)$ and applying Eq. 47

$$h_s(t) = \sum_{i=1}^{n} h_i(t) \qquad (99b)$$

Reliability Analysis of Systems

- Series Systems (cont'd)
  - For the case of the series system composed of identical components with CHRFs $H_c(t)$ and hazard rates $h_c(t)$, Eqs. 99a and 99b are reduced respectively to

$$H_s(t) = n\, H_c(t) \qquad (100a)$$

$$h_s(t) = n\, h_c(t) \qquad (100b)$$

Reliability Analysis of Systems

- Series Systems (cont'd)
  - Thus, the hazard functions for a series system can be easily evaluated based on the hazard functions of the system's components.
  - An examination of Eqs. 99 and 100 reveals that the series system composed of components having increasing hazard (failure) rate, has an increasing failure rate, which is illustrated by the following example (Example 21):

Reliability Analysis of Systems

- **Example 21:** Assessing the Hazard Function of a Series System of Three Identical Components
  - In this example, three identical components with the same hazard function are used to develop the system hazard function.
  - The component hazard functions is given by

$$H_c(t) = 0.262649 - 0.013915t + 0.000185t^2$$

  - and

$$h_c(t) = -0.013915 + 0.000370t$$

t in years

Reliability Analysis of Systems

- Example 21 (cont'd)
  - Applying Equations 99a and 99b with n = 3, the following expressions can be obtained:

$$H_s(t) = 0.787947 - 0.041745t + 0.000555t^2$$

and

$$h_s(t) = - 0.041745 + 0.001110t$$

  - The resulting hazard functions are given in Table 23 and Figures 17a and 17b

Reliability Analysis of Systems

Table 23. Hazard (Failure) Rate and Cumulative Hazard Rate Functions for a Series System of Three Identical Components of Example 21

Year	Time to Failure, Years	Hazard Rate Function	Cumulative Hazard Rate Function
1980	43	0.005985	0.019107
⋮	⋮	⋮	⋮
2007	70	0.035955	0.585297
2008	71	0.037065	0.621807
2009	72	0.038175	0.659427
2010	73	0.039285	0.698157

Reliability Analysis of Systems

- Example 21 (cont'd)

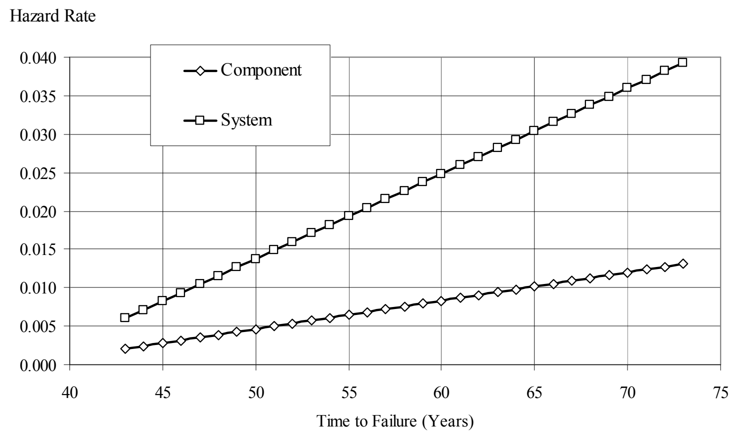

Figure 17a. Hazard (Failure) Rate Function (HRF) for a Series System of Three Identical Components of Example 21

Reliability Analysis of Systems

- Example 21 (cont'd)

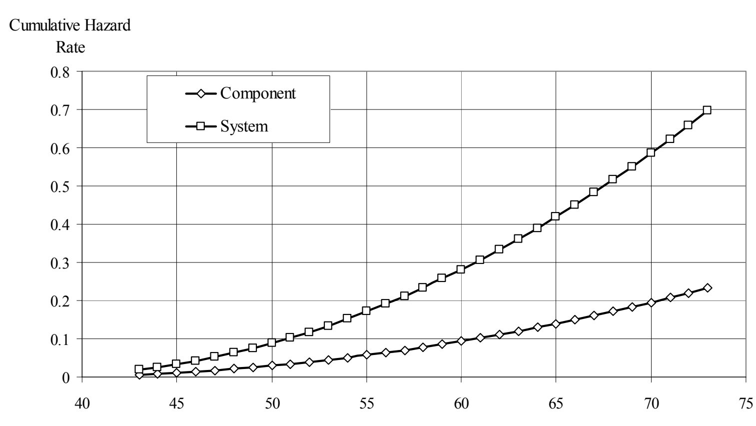

Figure 17b. Cumulative Hazard Rate (Failure) Function (CHRF) for a Series System of Three Identical Components of Example 21

Reliability Analysis of Systems

- **Example 22:** Assessing the Hazard Functions of a Series System of Four Different Components
  - The hazard rate functions for one component of this system are from Example 21.
  - In order to get the hazard rate functions for three other components, data from the *Emsworth Locks and Dams*, *Vertical Lift Gate Reliability Analysis* were used in a similar manner for three other components.

Reliability Analysis of Systems

- Example 22 (cont'd)
  - The failure data and survivor functions for these components are given in Tables 24, 25, and 26.
  - The parameters of the hazard rate functions based on Eqs. 60a and 60b model obtained for these components are given in Table 27.
  - The parameters of the hazard rate functions of the series system composed of these components were obtained using Eqs. 99a and 99b as given in Table 27.

Reliability Analysis of Systems

- Example 22 (cont'd)

Table 24. Data and Empirical Survivor Function, $S_n(t)$, for Component 2 for Example 22

Year	TTF (Years)	Number of Failures	Survivor Function
1937	0	0	1.000000
⋮	⋮	⋮	⋮
2001	64	184	0.855350
2002	65	189	0.845900
2003	66	190	0.836400
2004	67	193	0.826750

Reliability Analysis of Systems

- Example 22 (cont'd)

Table 25. Data and Empirical Survivor Function, $S_n(t)$, for Component 3 for Example 22

Year	TTF (Years)	Number of Failures	Survivor Function
1937	0	0	1.000000
⋮	⋮	⋮	⋮
2001	64	174	0.871750
2002	65	176	0.862950
2003	66	181	0.853900
2004	67	182	0.844800

Reliability Analysis of Systems

- Example 22 (cont'd)

Table 26. Data and Empirical Survivor Function, $S_n(t)$, for Component 4 for Example 22

Year	TTF (Years)	Number of Failures	Survivor Function
1937	0	0	1.000000
⋮	⋮	⋮	⋮
2001	64	195	0.846850
2002	65	198	0.836950
2003	66	202	0.826850
2004	67	209	0.816400

Reliability Analysis of Systems

- Example 22 (cont'd)

Table 27. Parameters of Hazard Rate Functions for Four Components and the Series System for Example 22

Component Number or System	Parameter a_0	Parameter a_1 (1/Year)	Parameter a_2 (1/Year^2)
Component 1	0.262649	-0.013915	0.000185
Component 2	0.261022	-0.014371	0.000199
Component 3	0.264099	-0.014097	0.000189
Component 4	0.281940	-0.015469	0.000213
Series System	1.069710	-0.057852	0.000786

Reliability Analysis of Systems

- Example 22 (cont'd)
  - Based on the parameter estimates for the series system, and applying Eqs. 99 and 100, the hazard rate functions can be estimated by algebraically summing up the component hazard functions.
  - The resulting system functions are

$$H_s(t) = 1.069710 - 0.057852t + 0.000786t^2$$

$$h_s(t) = - 0.057852 + 0.001572t$$

  - Figures 18a and 18b show the respective hazard rate functions.

Reliability Analysis of Systems

- Example 22 (cont'd)

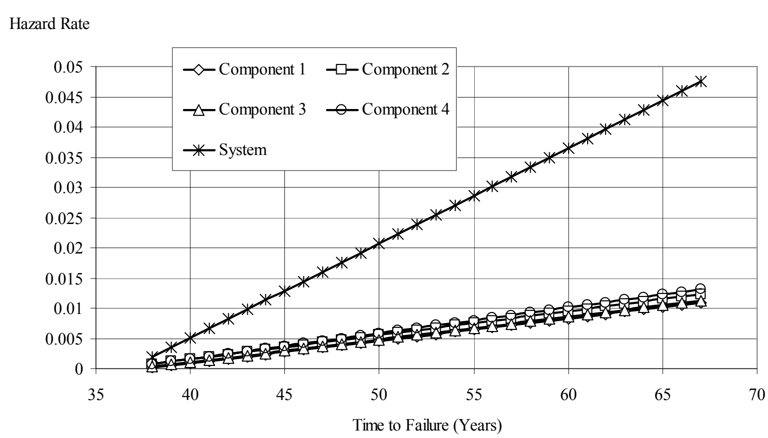

Figure 18a. Hazard Rate Functions (HRF) for Series System of Four Different Components of Example 22

Reliability Analysis of Systems

- Example 22 (cont'd)

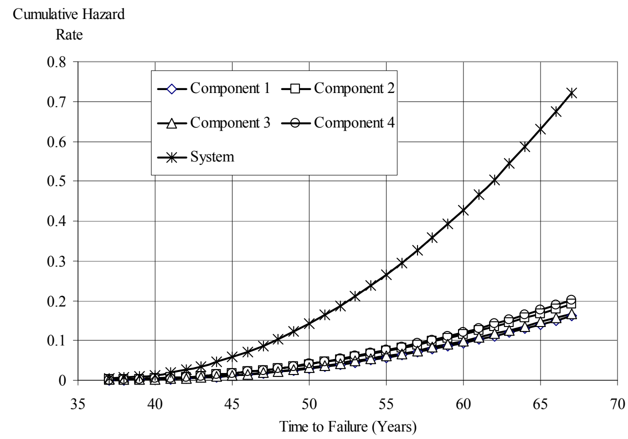

Figure 18b. Cumulative Hazard (Failure) Rate Functions (CHRF) for Series System of Four Different Components of Example 22

Reliability Analysis of Systems

- Parallel Systems
  - Figure 19 depicts an example of the EBD for a parallel system consisting of three components.
  - The reliability function of a parallel system composed of n components, $R_s(t)$, is given by

$$R_S(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t)) \qquad (101)$$

where $R_i(t)$ is the reliability function of i[th] component.

Reliability Analysis of Systems

- Parallel Systems (cont'd)

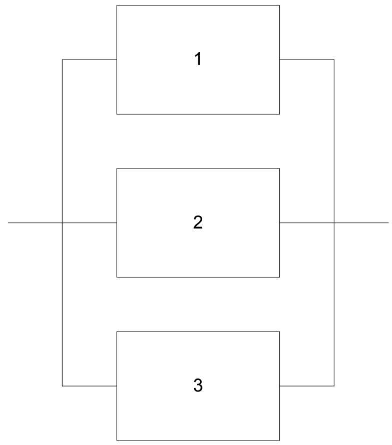

Figure 19. Parallel System Composed of Three Components

Reliability Analysis of Systems

- Parallel Systems (cont'd)
  - If a parallel system is composed of identical components with reliability functions, $R_c(t)$, Eq. 101 is reduced to

$$R_s(t) = 1 - (1 - R_c(t))^n \qquad (102)$$

  - Compared with a series system composed of the same components, the respective parallel system is always more reliable.
  - A parallel system is an example of a *redundant* system.

Reliability Analysis of Systems

- Parallel Systems (cont'd)
  - Relationship between parallel system cumulative hazard rate function, $H_s(t)$, (CHRF) and the reliability functions of its components, $R_i(t)$, can be written as

$$H_s(t) = -\ln(1 - \prod_{i=1}^{n}(1 - R_i(t))) \qquad (103a)$$

  - By taking the derivative of $H_s(t)$ and using Eq. 47, we obtain

$$h_s(t) = -\frac{\sum_{j=1}^{n}\left[\prod_{i=1, i\neq j}^{n}(1 - R_i(t))\frac{dR_j(t)}{dt}\right]}{R_s(t)} \qquad (103b)$$

Reliability Analysis of Systems

- Parallel Systems (cont'd)
  - For example, for n = 3, Eq. 103b takes on the following form:

$$h_s(t) = -\frac{(1-R_2(t))(1-R_3(t))\frac{dR_1(t)}{dt} + (1-R_1(t))(1-R_3(t))\frac{dR_2(t)}{dt} + (1-R_1(t))(1-R_2(t))\frac{dR_3(t)}{dt}}{1-(1-R_1(t))(1-R_2(t))(1-R_3(t))}$$

  - For practical problems, it might be better to apply numerical differentiation of Eq. 103a, instead of directly using Eq. 103b.

Reliability Analysis of Systems

- Parallel Systems (cont'd)
  - For the case of a parallel system composed of identical components with reliability functions $R_c(t)$, Eq. 103a and 103b are reduced to

$$H_s(t) = -\ln(1-(1-R_c(t))^n) \qquad (104a)$$

$$h_s(t) = -\frac{n(1-R_c(t))^{n-1}\dfrac{dR_c(t)}{dt}}{1-(1-R_c(t))^n} \qquad (104b)$$

Reliability Analysis of Systems

- **Example 23:** Assessing the Hazard Function of a Parallel System of Three Identical Components
  - A parallel system composed of the same identical components as were used in Example 21 is used to demonstrate the assessment of the system hazard functions.
  - Thus, for each component the hazard functions are

$$H_c(t) = 0.262649 - 0.013915t + 0.000185t^2$$

t in years

$$h_c(t) = -\ 0.013915 + 0.000370t$$

Reliability Analysis of Systems

- Example 23 (cont'd)
  - Applying Equation 45, the component reliability function is given by

$$R_c(t) = \exp(-(0.262649 - 0.013915t + 0.000185t^2))$$

  - In order to calculate system CHRF, $H_s(t)$, Eq. 104a can be used with n = 3

$$\frac{dR_c}{dt} = -(R_c(t))h_c(t)$$

  - The resulting hazard functions are given in Table 28 and illustrated by Figures 20a and 20b.

Reliability Analysis of Systems

- Example 23 (cont'd)

Table 28. Hazard Rate Functions for Parallel System Composed of Three Identical Components of Example 23

Year	Time to Failure (Years)	Hazard Rate Function	Cumulative Hazard Rate Function
1975	38	3.41962E-10	1.05647E-09
⋮	⋮	⋮	⋮
2001	64	0.000380821	0.001807317
2002	65	0.000449465	0.002223232
2003	66	0.000526673	0.002712222
2004	67	0.000612993	0.003283142

Reliability Analysis of Systems

- Example 23 (cont'd)

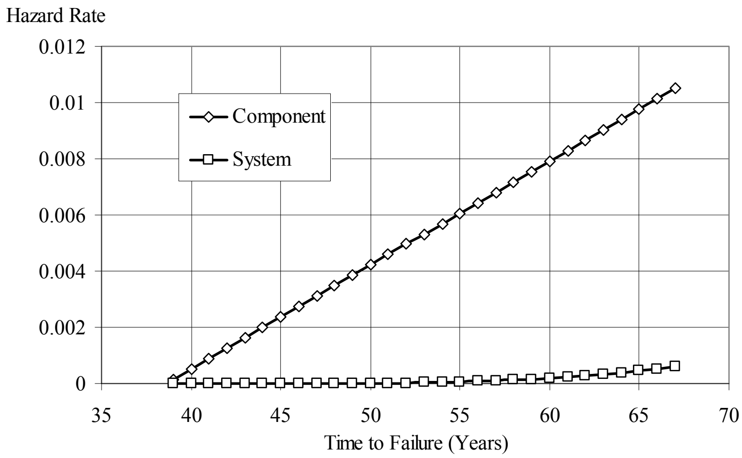

Figure 20a. Hazard (Failure) Rate Function (HRF) for Parallel System of Three Identical Components of Example 23

Reliability Analysis of Systems

- Example 23 (cont'd)

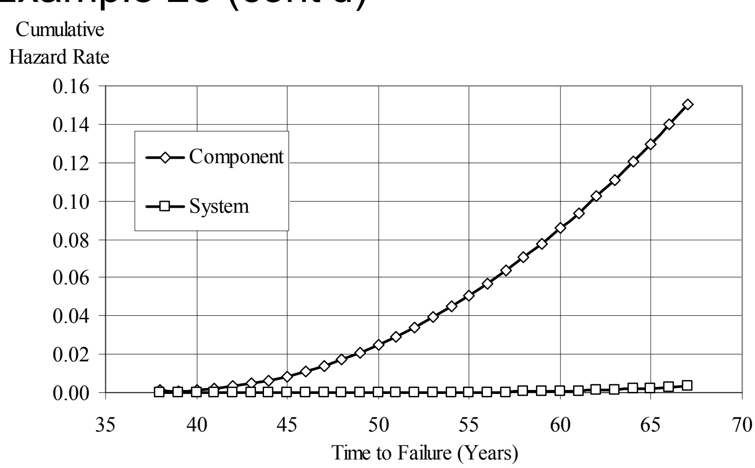

Figure 20b. Cumulative Hazard Rate Function (CHRF) for Parallel System of Three Identical Components of Example 23

Reliability Analysis of Systems

- **Example 24:** Assessing the Hazard Functions of a Parallel System of Four Different Components
  - The parallel system composed of the four different components used in Example 22 (shown in Table 27) is used in this example to demonstrate the case of components in parallel
  - The system CHRF, Hs(t), can be evaluated using Eqs. 103a and 103b.
  - The reliability functions of the system's components $R_i(t)$, can be determined using Eq. 45.

Reliability Analysis of Systems

- Example 24 (cont'd)
  - Instead of using Eq. 103b, the hazard (failure) rate function can be calculated using the following numerical approximation for the derivative of Eq. 10:

$$h_s(t_i) = \frac{H_s(t_i) - H_s(t_{i-1})}{t_i - t_{i-1}}$$

  - where t_i ($i = 1, 2, ..., n$) are successive times at which H_s is evaluated.

Reliability Analysis of Systems

- Example 24 (cont'd)
  - For the data used in report, the difference t_i - t_{i-1} is equal to one year.
  - The resulting hazard functions are given in Table 29 and shown in Figures 21a and 21b.

Reliability Analysis of Systems

- Example 24 (cont'd)

Table 29. Hazard Rate Functions for a Parallel System Composed of Four Different Components of Example 24

Year	Time to Failure, Years	Hazard Rate Function	Cumulative Hazard Rate Function
1975	38	2.50140E-12	5.20173E-12
⋮	⋮	⋮	⋮
2001	64	8.22737E-05	3.39908E-04
2002	65	1.03504E-04	4.43412E-04
2003	66	1.28933E-04	5.72345E-04
2004	67	1.59129E-04	7.31474E-04

Reliability Analysis of Systems

- Example 24 (cont'd)

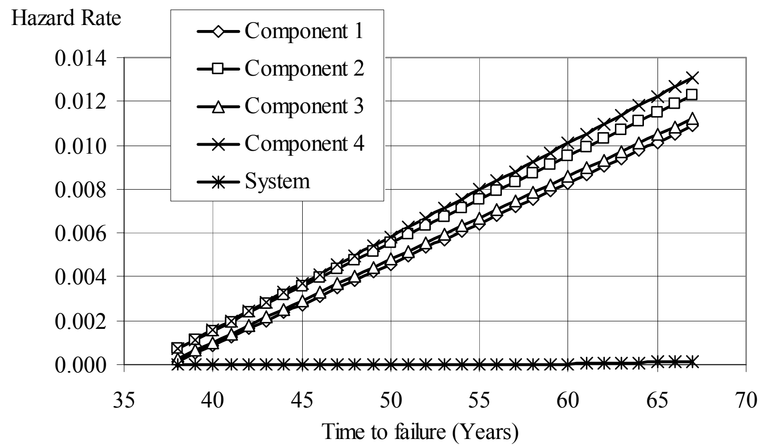

Figure 21a. Hazard (Failure) Rate Function (HRF) for Parallel System of Four Different Components of Example 24

Reliability Analysis of Systems

- Example 24 (cont'd)

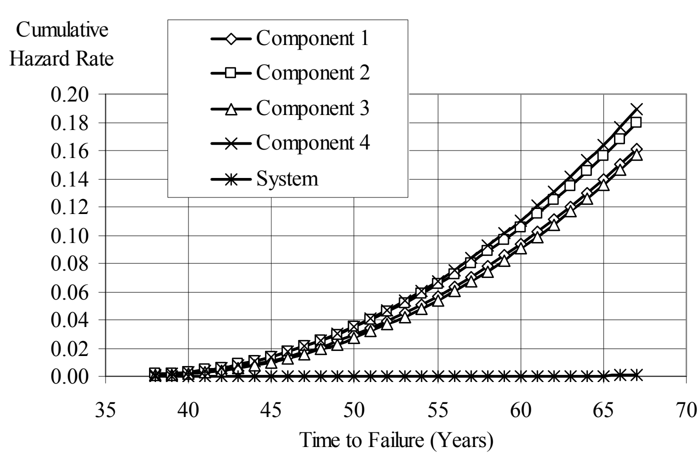

Figure 21b. Cumulative Hazard Rate Function (CHRF) for Parallel System of Four Different Components of Example 24

Reliability Analysis of Systems

- Series-Parallel Systems
  - Some systems, from the reliability standpoint, can be represented as a series structure of k parallel structures.
  - An example reliability block diagram of such system is shown in Figure 22.
  - Such systems are also called *series-parallel* systems.
  - These systems are redundant and have alternate loads (or demand) paths.

Reliability Analysis of Systems

- Series-Parallel Systems (cont'd)

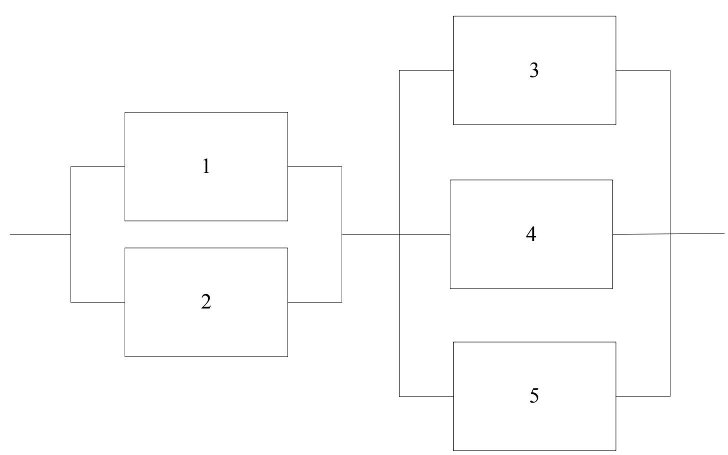

Figure 22. Series Structure of Two Parallel Structures

Reliability Analysis of Systems

- Series-Parallel Systems (cont'd)
  - the system in Figure 22 can be represented as a series system of two subsystems, called herein Subsystem 1 and Subsystem 2.
  - Subsystem 1 is composed of components 1 and 2, connected in parallel, and Subsystem 2 is composed of components 3, 4, and 5 also connected in parallel.
  - Hence, the equivalent structure of the system considered can be represented by the RBD given by Figure 23.

Reliability Analysis of Systems

- Series-Parallel Systems (cont'd)

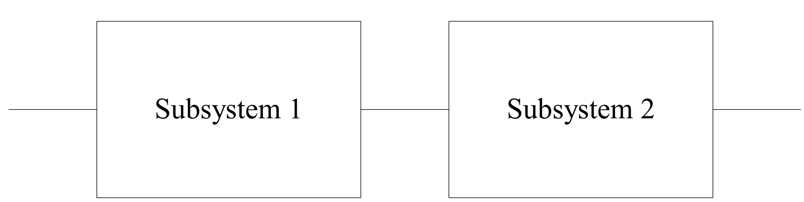

Figure 23. . A System of Composing Components Equivalent to the System Given in Figure 22

Reliability Analysis of Systems

- Series-Parallel Systems (cont'd)

Steps to compute the reliability and hazard functions of the system:

1. Calculate the reliability functions of Subsystems 1 and 2, using Eq. 101 (for parallel systems);
2. Based on the results from the first step, calculate the reliability function of the series system composed of Subsystems 1 and 2 using Eq. 97 (for series systems); and
3. 3. Using basic relationships between the reliability function and hazard functions, i.e., Eqs. 9 and 10, calculate the cumulative hazard rate function and hazard (failure) rate function for the system of interest represented in Figure 22.

Reliability Analysis of Systems

- Series-Parallel Systems (cont'd)
  - If one is interested in assessing the hazard functions only, the problem can be solved in the following way:
    1. Calculate the hazard functions for each subsystem as described in Section 4.4.3 for parallel systems; and
    2. Calculate the system hazard rate function as the hazard rate functions of the series system composed of the subsystems as components of the series system.

Reliability Analysis of Systems

- **Example 25:** Assessing the Hazard Functions of a Series-Parallel System
  - A series-parallel system consisting of two identical subsystems is considered.
  - Each subsystem is composed of the four components connected in parallel that were considered in Example 24.
  - The hazard functions of each subsystem are exactly the same as the respective hazard functions $H_s(t)$ and $h_s(t)$ obtained in Example 24.

Reliability Analysis of Systems

- Example 25 (cont'd)
  - According to Eqs. 99 and 100, the hazard function for the series-parallel system can be based on the hazard functions $H_s(t)$ and $h_s(t)$ from Example 24.
  - The values of these functions are given in Table 30, and they are depicted in Figures 24a and 24b.

Reliability Analysis of Systems

- Example 25 (cont'd)

Table 30. Assessing the Hazard Functions of a Series-parallel System of Example 4-25

Year	Time to Failure (Years)	Hazard Rate Function	Cumulative Hazard Rate Function
1975	38	5.00289E-12	1.040346E-11
⋮	⋮	⋮	⋮
2001	64	1.64547E-04	6.798169E-04
2002	65	2.07007E-04	8.868240E-04
2003	66	2.57865E-04	1.144689E-03
2004	67	3.18258E-04	1.462947E-03

Reliability Analysis of Systems

- Example 25 (cont'd)

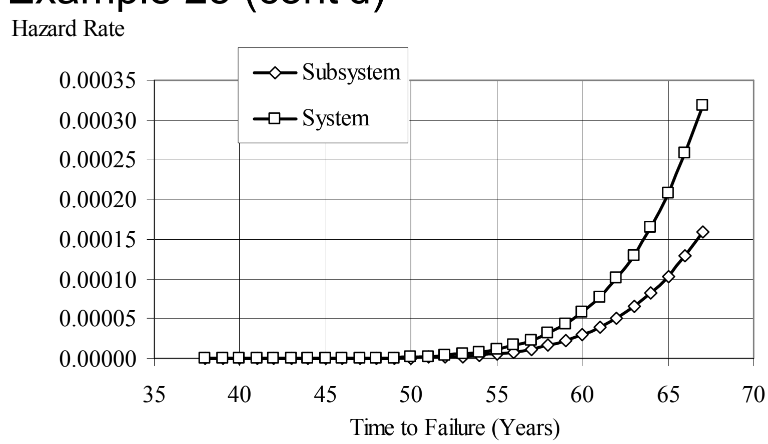

Figure 24a. Hazard Rate Function (HRF) for a Series-Parallel System of Example 25

Reliability Analysis of Systems

- Example 25 (cont'd)

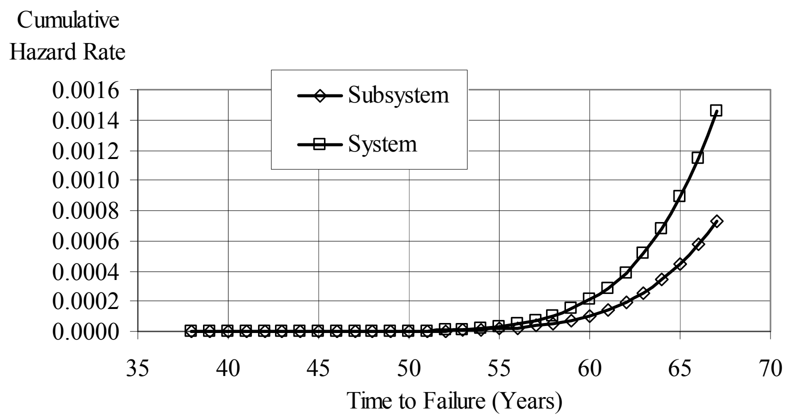

Figure 24b. Cumulative Hazard Rate Function (CHRF) for a Series-Parallel System of Example 25

Reliability Analysis of Systems

- *k*-out-of-*n* Systems
  - Such a system has *n* parallel components, however, at least *k* component must be functioning, if the system is to continue operating.
  - An example of this type of redundant systems is cables for a bridge, where a certain minimum number of cables are necessary to support the structure.

Reliability Analysis of Systems

- *k*-out-of-*n* Systems (cont'd)
  - Another example of *k*-out-of-*n* systems is a three-engine airplane, which can stay in the air if and only if at least two of its three engines are functioning, i.e., the plane can be modeled by a 2-out-of-3 system.
  - The respective reliability block diagram for a 2-out-of-3 structure is given in Figure 25.
  - The reliability block diagram (RBD) of Figure 25 has more "components" than the real system, which is why the technique of system reliability evaluation, considered in the previous sections are not applicable to *k*-out-of-*n* systems.

Reliability Analysis of Systems

- *k*-out-of-*n* Systems (cont'd)

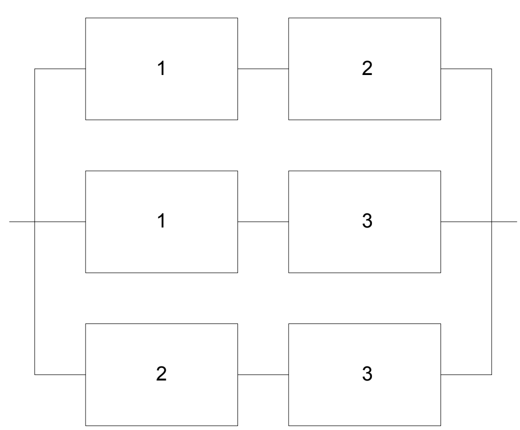

Figure 25. Two-Out-of-Three System

Reliability Analysis of Systems

- *k*-out-of-*n* Systems (cont'd)
  - In the engineering practice, parallel systems and k out of n systems are usually composed of identical components.
  - The reliability function of the *k*-out-of-*n* system, R_s, is given by:

$$R_S(t) = \sum_{i=k}^{n} \binom{n}{i} \left(R_C(t)\right)^i \left(1 - R_C(t)\right)^{n-i} \qquad (105)$$

Reliability Analysis of Systems

- *k*-out-of-*n* Systems (cont'd)
  - Relationship between the *k*-out-of-*n* system cumulative hazard rate function, $H_s(t)$, (CHRF) and the reliability function of its (identical) components, $R_c(t)$, can be written:

$$H_S(t) = -\ln\left(\sum_{i=k}^{n} \binom{n}{i} \left(R_C(t)\right)^i \left(1 - R_C(t)\right)^{n-i} \right) \qquad (106)$$

Reliability Analysis of Systems

- *k*-out-of-*n* Systems (cont'd)
  - In order to assess the respective system hazard (failure) rate function, hs(t), the basic relationship, i.e., Eq. 10, between the hazard (failure) rate function and the cumulative hazard rate function in the form of Eq. 106 needs to be used.
  - Due to rather complex form of Eq. 106, numerical differentiation is recommended for practical problems.

Reliability Analysis of Systems

- **<u>Example 26:</u>** Assessing the Hazard Functions of a Two-out-of-Three System of Identical Components
  - A 2-out-of-3 system composed of identical components having a reliability function from the USACE demonstrative Emsworth Locks and Dams, Vertical Lift Gate Reliability Analysis is examined herein.
  - The component reliability function is given by

$$R_c(t) = \exp(-0.262649 + 0.013915t - 0.000185t^2)$$

t in years

Reliability Analysis of Systems

- Example 26 (cont'd)
  - Equation 106 can be used to assess the 2-out-of-3 system cumulative hazard rate function, $H_s(t)$, which takes the following form

$$H_s(t) = -\ln\left(\sum_{i=2}^{3}\binom{3}{i}\left(R_c(t)\right)^i\left(1-R_c(t)\right)^{n-i}\right)$$

  - The function above can be calculated using function **BINOMDIST** in Microsoft-Excel.

Reliability Analysis of Systems

- Example 26 (cont'd)
  - For this example, the hazard (failure) rate function can be calculated using the same approximation, as in Example 24.
  - The results of the hazard functions calculations are given in Table 31 and in Figures 26a and 26b.

Reliability Analysis of Systems

- Example 26 (cont'd)

Table 31. Hazard (Failure) Rate and Cumulative Hazard Rate Functions for of Two-out-of-Three System of Example 26

Year	Time to Failure (Years)	Hazard Rate Function	Cumulative Hazard Rate Function
1980	43	5.85E-05	1.20E-04
2007	70	9.07E-03	8.68E-02
2008	71	9.75E-03	9.65E-02
2009	72	1.04E-02	1.07E-01
2010	73	1.12E-02	1.18E-01

Reliability Analysis of Systems

- Example 26 (cont'd)

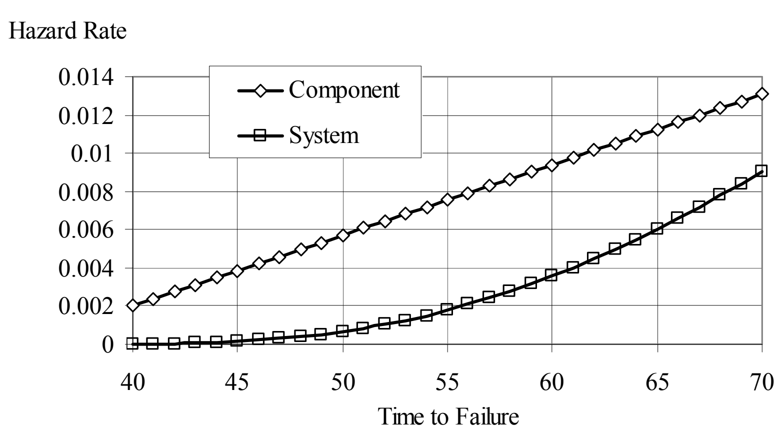

Figure 26a. Hazard (Failure) Rate Function (HRF) of a Two-out-of-Three System of Identical Components of Example 26

Reliability Analysis of Systems

- Example 26 (cont'd)

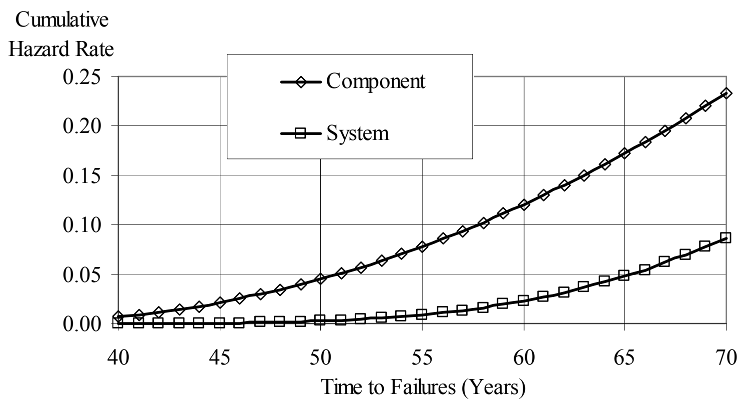

Figure 26b. Cumulative Hazard Rate Function (CHRF) of a Two-out-of-Three System of Identical Components of Example 26

Reliability Analysis of Systems

- **Example 27:** Three-Component Series System as a Three-out-of-Three System, and Three-Component Parallel System as One-out-of-Three System
  - The difference between the 2-out-of-3 system on one hand, and the parallel and series systems composed of the same 3 identical components on the other hand is explored in this example.

Reliability Analysis of Systems

- Example 27 (cont'd)
  - The series system can be treated as a 3-out-of-3 system, and the parallel system can be treated as a 1-out-of-3 system.
  - The respective hazard functions for these systems are shown in Figures 27a, and 27b.
  - The figures clearly show that the series (3-out-of-3) system is the least reliable, the parallel (1-out-of-3) system is the most reliable.
  - Meanwhile the hazard rate functions of 2-out-of-3 system is somewhere between the hazard rate functions of the series (3-out-of-3) system and the parallel (1-out-of-3) system.

Reliability Analysis of Systems

- Example 27 (cont'd)

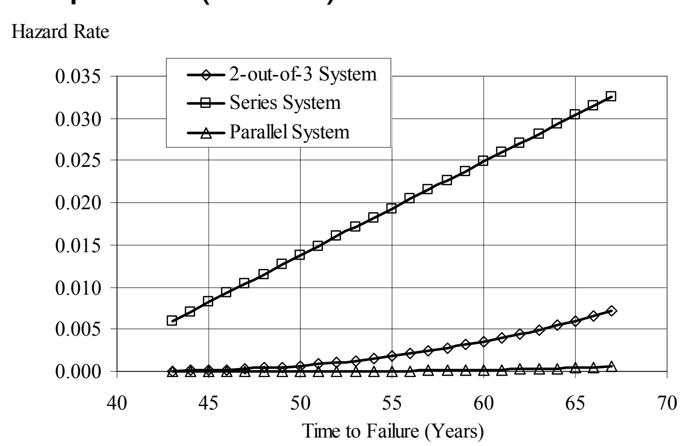

Figure 27a. Hazard Rate Functions (HRF) of a Two-out-of-Three System, a Parallel System, and a Series System Composed of Three Identical Components of Example 27

Reliability Analysis of Systems

- Example 27 (cont'd)

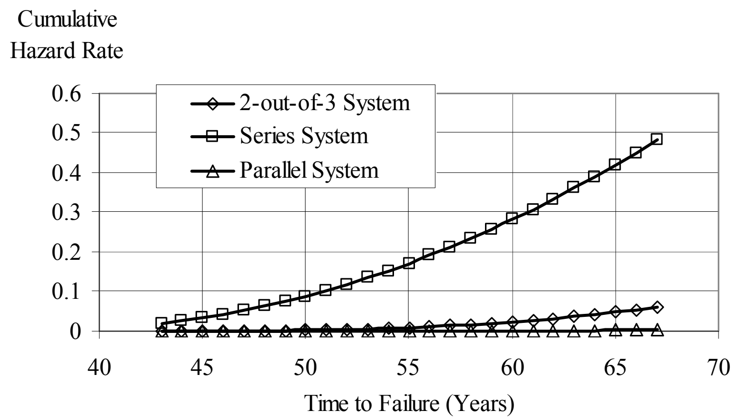

Figure 27b. Cumulative Hazard Rate Functions (CHRF) of a Two-out-of-Three System, a Parallel System, and a Series System Composed of Three Identical Components of Example 27

Homework Assignment #4

Problems:
4.x
4.x
4.x
4.x
4.x