CHAPTER

4b

Risk Analysis in Engineering and Economics

RELIABILITY ASSESSMENT

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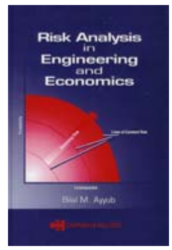

Risk Analysis for Engineering

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Empirical Reliability Analysis Using Life Data

- Availability
  - If the time to failure is characterized by its mean, called mean time to failure (MTTF), and the time to repair is characterized by its mean, called mean time to repair (MTTR), a definition of this probability of finding a given product in a functioning state can be given by the following ratio for availability (A):

$$A = \frac{MTTF}{MTTF + MTTR} \qquad (35)$$

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions
  - As a random variable, the time to failure (TTF or *T* for short) is completely defined by its *reliability function*, *R*(*t*).
  - The *reliability function* is defined as the probability that a unit or a component does not fail in the time interval (0,*t*] or, equivalently, the probability that the unit or the component survives the time interval (0, *t*], under a specified environment.

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - The probability part of this definition of the TTF can be expressed using the reliability function *R*(*t*) as follows:

$$R(t) = \Pr\,(T > t) \qquad (35)$$

Pr = probability
T = time to failure
t = any time period

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - The reliability function is also called the *survivor* (or *survivorship*) function.
  - Another function, that can completely define any random variable (e.g., time to failure as well as time to repair) is the *cumulative distribution* function. This function is given as

$$F(t) = 1 - R(t) = \Pr(T \leq t) \qquad (36)$$

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - The CDF is the probability that the product does not survive the time interval (0, *t*].
  - Assuming the TTF as a random variable to be a continuous positively defined, and *F*(*t*) to be differentiable, the CDF can be written as

$$F(t) = \int_0^t f(x)\,dx \quad \text{for } t > 0 \qquad (37a)$$

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - Exponential Distribution
    - The exponential distribution has a reliability function $R(t)$ as given by

$$R(t) = \exp(-\lambda t) \tag{38}$$

λ = failure rate = constant

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - Weibull Distribution
    - The reliability function of the two-parameter Weibull distribution is

$$R(t) = \exp[-(t/\alpha)^{\beta}] \tag{39}$$

α = scale parameter
β = shape parameter

Empirical Reliability Analysis Using Life Data

- Reliability, Failure Rates, and Hazard Functions (cont'd)
  - Lognormal Distribution
    - The reliability function of the lognormal distribution is given by

$$R(t) = 1 - \Phi\left(\frac{\ln(t) - \mu}{\sigma}\right) = \Phi\left(-\frac{\ln(t) - \mu}{\sigma}\right) \quad (40)$$

μ = log mean
σ = log standard deviation
Φ(.) = standard normal cumulative distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{x^2}{2}\right) dx \quad (41)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions
  - The conditional probability

$$\Pr(t < T \leq t + \Delta t \mid T > t)$$

is the failure probability of a product unit in the time interval $(t, t + \Delta t]$, with the condition that the unit is functioning at time t, for small Δt.
  - This conditional probability can be used as a basis for defining the hazard function for the unit by expressing the conditional probability as

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)

$$\Pr(t < T \le t + \Delta t \mid T > t) = \frac{f(t)}{R(t)}\Delta t = h(t)\Delta t \quad (42)$$

$$h(t) = \frac{f(t)}{R(t)} \quad (43)$$

$h(t)$ = hazard (or failure) rate function

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - The CDF, $F(t)$, for the time to failure, $F(t)$, and the reliability function, $R(t)$, can always be expressed in terms of the so-called cumulative hazard rate function (CHRF), $H(t)$, as follows:

$$F(t) = 1 - \exp(-H(t)) \quad (44)$$

$$R(t) = \exp[-H(t)] \quad (45)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - Based on Eq. 45, the CHRF can be expressed through the respective reliability function as

$$H(t) = -\ln[R(t)] \quad (46)$$

  - It can be shown that the cumulative hazard rate function and the hazard (failure) rate function are related to each other as

$$h(t) = \frac{dH(t)}{dt} \quad (47)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - The cumulative hazard rate function and its estimates must satisfy the following conditions:

$$H(0) = 0 \quad (48a)$$

$$\lim_{t \to \infty} \left(H(t)\right) = \infty \quad (48b)$$

$H(t)$ = non-decreasing function that can be expressed as

$$\frac{dH(t)}{dt} = h(t) \geq 0 \quad (48c)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - For the exponential distribution, the hazard (failure) rate function is constant, and is given by

$$h(t) = \lambda \qquad (49)$$

and the exponential cumulative hazard rate function is

$$H(t) = \lambda t \qquad (50)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - The Weibull hazard (failure) rate function is a power law function, which can be written as

$$h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \qquad (51)$$

  - and the respective Weibull cumulative hazard rate function is

$$H(t) = (t/\alpha)^{\beta} \qquad (52)$$

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - For the lognormal distribution, the cumulative hazard (failure) rate function can be obtained, using Eqs. 46 and 40, as

$$H(t) = -\ln\left(\Phi\left(-\frac{\ln(t)-\mu}{\sigma}\right)\right) \qquad (53)$$

μ = log mean
σ = log standard deviation
Φ(.) = standard normal cumulative distribution function

Empirical Reliability Analysis Using Life Data

- Hazard Functions (cont'd)
  - The lognormal hazard (failure) rate function can be obtained as the derivative of the corresponding CHRF:

$$\begin{aligned} h(t) &= \frac{dH(t)}{dt} \\ &= \frac{\frac{1}{t\sigma}\phi\left(\left(\frac{\mu-\ln(t)}{\sigma}\right)\right)}{\Phi\left(\left(\frac{\mu-\ln(t)}{\sigma}\right)\right)} \end{aligned} \qquad (54)$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - The best lifetime distribution for a given product is the one based on the probabilistic physical model of the product.
  - Unfortunately, such models might not be available.
  - Nevertheless, the choice of the appropriate distribution should not be absolutely arbitrary, and at least some physical requirements must be satisfied.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Complete Data, Without Censoring
    - If the available data are complete, i.e., without censoring, the following empirical reliability (survivor) function, i.e., estimate of the reliability function, can be used:

$$S_n(t) = \begin{cases} 1 & 0 \le t < t_1 \\ \dfrac{n-i}{n} & t_i \le t < t_{i+1} \text{ and } i = 1, 2, \ldots, n-1 \\ 0 & t_n \le t < \infty \end{cases} \qquad (55)$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models

t_i = the i^th^ failure time denoted according to their ordered values (order statistics) as $t_1 < t_2 < \ldots < t_k$
k = the number of failures, and n is the sample size

  – In the case of complete data with distinct failures, $k = n$.
  – The estimate can also be applied to the Type I and II right-censored data.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  – In the case of ***Type I*** censoring, the time interval of $S_n(t)$ estimation is $(0, T]$, where $T = t_0$ is the test (or observation) duration.
  – In the case of ***Type II*** censoring, the respective time interval is $(0, t_r]$, where t_r is the largest observed failure time.
  – This commonly-used estimate, $S_n(t)$, is called the empirical survivor function.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Complete Data, Without Censoring (cont'd)
    - Based on 55, an estimate of the CDF of TTF can be obtained as

$$F_n(t) = 1 - S_n(t) \tag{56}$$

F_n(t) = estimate of the CDF of time to failure

Empirical Reliability Analysis Using Life Data

- Example 5: Single Failure Mode, Small Sample Without Censoring
  - The single failure mode, non-censored data presented in Example 2 are used to illustrate the estimation of an empirical reliability function using Eq. 55.
  - The sample size n in this case is 19.
  - The TTFs and the results of calculations of the empirical survivor function $S_n(t)$ are given in Table 4.

Empirical Reliability Analysis Using Life Data

Time Order Number	*TTF* (Years)	Empirical Survivor Function
0	0	19/19 = 1
1	26	18/19 = 0.947368
2	27	17/19 = 0.894737
3	28	16/19 = 0.842105
4	29	15/19 = 0.789474
5	30	14/19 = 140.736842
6	31	13/19 = 0.684211
7	32	12/19 = 0.631579
8	33	11/19 = 0.578947
9	34	10/19 = 0.526316
10	35	9/19 = 0.473684
11	36	8/19 = 0.421053
12	37	7/19 = 0.368421
13	38	6/19 = 0.315789
14	39	5/19 = 0.263158
15	40	4/19 = 0.210526
16	42	3/19 = 0.157895
17	43	2/19 = 0.105263
18	50	1/19 = 0.052632
19	56	0/19 = 0

Table 4
Empirical Survivor Function $S_n(t)$, Based on Data of Example 2

Empirical Reliability Analysis Using Life Data

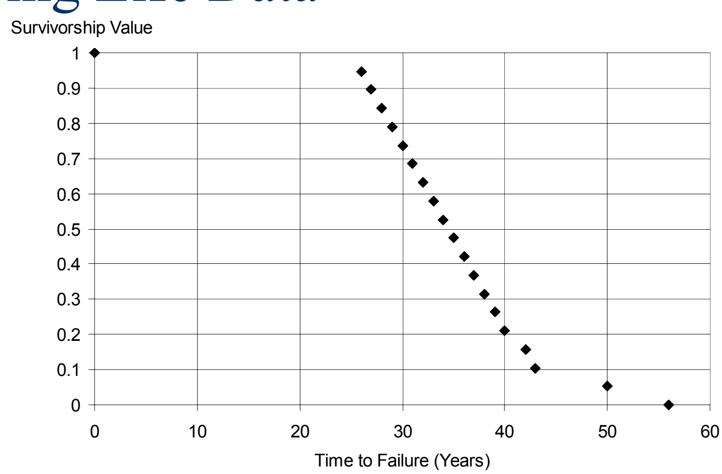


Figure 6. . Survivorship Function for Single Failure Mode Without Censoring of Example 5

Empirical Reliability Analysis Using Life Data

- Example 6: Single Failure Mode, Small Sample, Type I Right Censored Data
  - Equation 55 can be applied to Type I and II right censored data as was previously stated, which is illustrated in this example.
  - The data for this example are given in Table 1 as based on single failure mode, Type I right-censored data.

Empirical Reliability Analysis Using Life Data

- Example 6 (cont'd)

Table 1. Example of Type I Right Censored Data (in Years) for Equipment

Time Order Number	1	2	3	4	5	6	7	8	9	10	11	12
Time (Years)	7	14	15	18	31	37	40	46	51	51	51	51
TTF or TTC	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTC	TTC	TTC	TTC

TTF = time to failure, and TTC = time to censoring

- The TTFs and the calculation results of the empirical survivor function based on Eq. 55 are given in Table 5.
- The sample size n case is 12.

Empirical Reliability Analysis Using Life Data

- Example 6 (cont'd)

Table 5
Empirical Survivor Function, $S_n(t)$, Based on Data Given in Table 1

Time Order Number	Time to Failure, *TTF* (Years)	Time to Censoring, *TTF* (Years)	Empirical Survivor Function
0	0		1.000000
1	7		0.916667
2	14		0.833333
3	15		0.750000
4	18		0.666667
5	31		0.583333
6	37		0.500000
7	40		0.416667
8	46		0.333333
9		51	0.333333
10		51	0.333333
11		51	0.333333
12		51	0.333333

Empirical Reliability Analysis Using Life Data

- Example 6 (cont'd)
  - Censoring was performed at the end, i.e., without any censoring in between failures.
  - The empirical survivor function in the case of right censoring does not reach the zero value on the right, i.e., at the longest TTF observed.
  - The results are plotted in Figure 7 as individual points.

Empirical Reliability Analysis Using Life Data

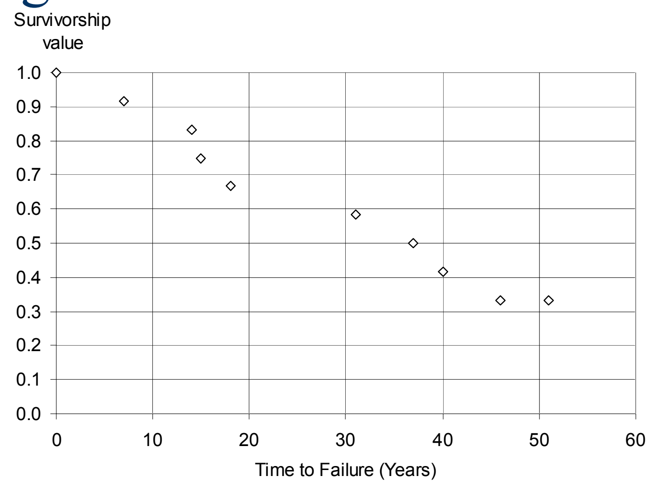


Figure 7. Survivorship Function for Single Failure Mode With Censoring of Example 6

Empirical Reliability Analysis Using Life Data

- Example 7: Single Failure Mode, Large Sample Data
  - The data in this example are based on Monte Carlo simulation.
  - The TTFs and the estimation results of the empirical survivor function based on Eq. 55 are given in Table 6.
  - The table shows only a portion of data since the simulation process was carried out for 20,000 simulation cycles.

Empirical Reliability Analysis Using Life Data

- Example 7 (cont'd)

Table 6. Example 7 Data and Empirical Survivor Function, $S_n(t)$

Year	*TTF* (Years)	Number of Failures	Survivor Function
1937	0	0	1.000000
⋮	⋮	⋮	⋮
2001	64	170	0.876900
2002	65	172	0.868300
2003	66	177	0.859450
2004	67	179	0.850500

Empirical Reliability Analysis Using Life Data

- Example 7 (cont'd)
  - The complete data set covers years from 1937 to 2060.
  - For example, the survivorship value at the year 1974 is computed as

$$\frac{(20{,}000-5)}{20{,}000} = 0.999750$$

  - The empirical survivorship values are shown in Figure 8.

Empirical Reliability Analysis Using Life Data

■ Example 7 (cont'd)

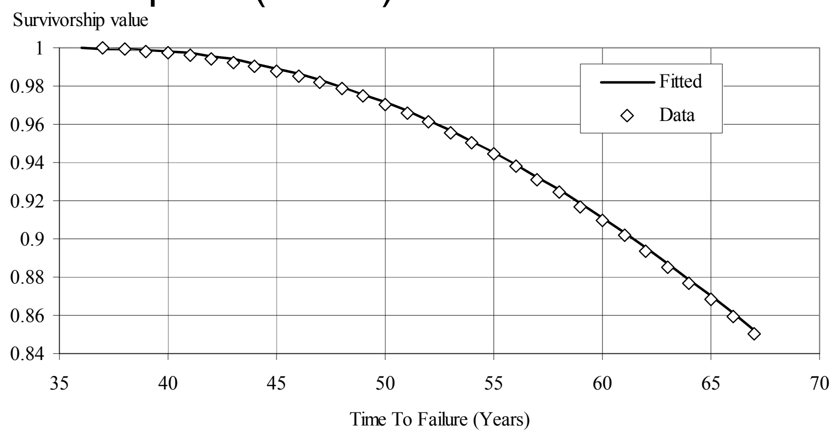


Figure 8. Empirical Survivor Function for Example 7

Empirical Reliability Analysis Using Life Data

■ Selection and Fitting Reliability Models

– Samples With Censoring

- In the this case, the Kaplan-Meier (or product-limit) estimation procedure can be applied to obtain the survivor function that accounts for both TTFs and TTCs.
- The Kaplan-Meier estimation procedure is based on a sample of n items, among which only k values are distinct failure times with r observed failures.
- Therefore, r minus k (i.e., r-k) repeated (non-distinct) failure times exist.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Samples With Censoring (cont'd)
    - The failure times are denoted similar to Eqs. 33a and 33b, according to their ordered values: $t_1 < t_2 < ... < t_k$, and t_0 is equal to zero, i.e., $t_0 = 0$.
    - The number of items under observation (censoring) just before t_j is denoted by n_j.
    - The number of failures at t_j is denoted by d_j. Then, the following relationship holds:

$$n_{j+1} = n_j - d_j \qquad (57)$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Samples With Censoring (cont'd)
    - Under these conditions, the product-limit estimate of the reliability function, $S_n(t)$, is given by

$$S_n(t) = \begin{cases} 1 & 0 \le t < t_1 \\ \prod_{j=1}^{i} \left(\dfrac{n_j - d_j}{n_j} \right) & t_i \le t < t_{i+1} \text{ for } i = 1,2,...,k-1 \\ 0 & t_k \le t < \infty \end{cases} \qquad (58)$$

t = time to failure of an equipment

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Samples With Censoring (cont'd)
    - For cases where d_j = 1, i.e., one failure at time t_j, Eq. 58 becomes

$$S_n(t) = \begin{cases} 1 & 0 \le t < t_1 \\ \prod_{j=1}^{i} \left(\dfrac{n_j - 1}{n_j} \right) & t_i \le t < t_{i+1} \text{ for } i = 1,2,\ldots,k-1 \\ 0 & t_k \le t < \infty \end{cases} \quad (59)$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Samples With Censoring (cont'd)
    - For uncensored (complete) samples with d_j = 1, the product-limit estimate coincides with the empirical $S_n(t)$ given by Eq. 55 as follows:

For $i = 1$:
$$S_n(t) = \prod_{j=1}^{1} \left(\frac{n_j - 1}{n_j} \right) = \left(\frac{n-1}{n} \right) = \left(\frac{n-1}{n} \right)$$

For $i = 2$:
$$S_n(t) = \prod_{j=1}^{2} \left(\frac{n_j - 1}{n_j} \right) = \left(\frac{n-1}{n} \right)\left(\frac{n-2}{n-1} \right) = \left(\frac{n-2}{n} \right)$$

For $i = 3$:
$$S_n(t) = \prod_{j=1}^{3} \left(\frac{n_j - 1}{n_j} \right) = \left(\frac{n-1}{n} \right)\left(\frac{n-2}{n-1} \right)\left(\frac{n-3}{n-2} \right) = \left(\frac{n-3}{n} \right)$$

$\vdots$ $\quad\vdots\quad$ $\vdots$

Therefore for any i:
$$S_n(t) = \prod_{j=1}^{i} \left(\frac{n_j - 1}{n_j} \right) = \left(\frac{n-i}{n} \right)$$

Empirical Reliability Analysis Using Life Data

- Example 8: A Small Sample with Two Failure Modes
  - In this example, life data consist of times to failure related to multiple failure modes (FMs).
  - The reliability function corresponding to each FM needs to be estimated using Eq. 58.
  - As an example, two FMs, i.e., FM1 and FM2, are considered herein.

Empirical Reliability Analysis Using Life Data

- Example 8 (cont'd)
  - Such TTF sample can be represented, for example, as follows:

$$t_1 \text{ (FM1)} \le t_2 \text{ (FM1)} \le t_3 \text{ (FM2)} \le t_4 \text{ (FM1)} \le \ldots \le t_k \text{ (FM2)}$$

  - For cases involving more than two FMs in a sample, the reliability function for a specific FMi can be estimated by treating the TTFs associated with failure modes other than FMi as times to censoring (TTC).

Empirical Reliability Analysis Using Life Data

- Example 8 (cont'd)
  - It should be noted that censoring means that an item survived up to the time of censoring and the item was removed from testing or service.
  - A sample of 12 TTFs associated with two failure modes, strength (FM1) and fatigue (FM2), are shown in Table 7a.
  - The calculations of the empirical survivor function based on Eq. 58 are given in Table 7a.

Empirical Reliability Analysis Using Life Data

Table 7a. Example 8 Small Sample Data and Respective Empirical Survivor Function for Failure Mode 1 $S_n(t)$

Time Order Number	Time to Failure (Years)	Number of Occurrences of Failure Mode 1 (Strength)	Number of Occurrences of Failure Mode 2 (Fatigue)	Empirical Survivor Function for Failure Mode1 (Strength)
0	0			1.000000
1	0.1	0	1	1.000000
2	1.1	0	1	1.000000
3	1.9	0	1	1.000000
4	6.2	0	1	1.000000
5	9.0	0	1	1.000000
6	11.7	0	1	1.000000
7	16.2	1	0	0.833333
8	21.3	0	1	0.833333
9	49.6	1	0	0.625000
10	51.0	1	0	0.416667
11	51.7	1	0	0.208333
12	68.3	1	0	0.000000

Empirical Reliability Analysis Using Life Data

- Example 8 (cont'd)
  - Table 7b provides the computational details of the empirical survivorship values for failure mode 1, with the sample size n = 12 and c_j = number of items censored at time j.
  - At time order 7 of Tables 7a and 7b,

    $S_n(16.2) = 1\text{-}1/6 = 0.8333$
  - Similarly at the time order number 9 of these tables,

    $S_n(49.6) = (1\text{-}1/6)(1\text{-}1/4) = 0.625$

Empirical Reliability Analysis Using Life Data

Table 7b. Example 8 Computational Details for Empirical Survivor Function for Failure Mode 1 $S_n(t)$

Time Order Number j	Time to Failure (Years) t_j	Number of Failures for Mode 1 d_j	Number of Censorings for Mode 1 c_j	$n_j = n - d_{j-1} - c_{j-1}$	$(1\text{-}d_j/n_j)$	Empirical Survivor Function for Mode1
0	0					1.000000
1	0.1	0	1	12		1.000000
2	1.1	0	1	11		1.000000
3	1.9	0	1	10		1.000000
4	6.2	0	1	9		1.000000
5	9.0	0	1	8		1.000000
6	11.7	0	1	7		1.000000
7	16.2	1	0	6	1-1/6	0.833333
8	21.3	0	1	5		0.833333
9	49.6	1	0	4	1-1/4	0.625000
10	51.0	1	0	3	1-1/3	0.416667
11	51.7	1	0	2	1-1/2	0.208333
12	68.3	1	0	1	0	0.000000

Empirical Reliability Analysis Using Life Data

- Example 9: Large Sample with Two Failure Modes
  - Two failure modes, strength (FM1) and fatigue (FM2), are simulated in this example.
  - A portion of these data related to one component is examined herein.
  - The full sample size is 20,000,
  - The TTFs and the results of calculations of the empirical survivor function based on Eq. 58 are given in Table 8.

Empirical Reliability Analysis Using Life Data

- Example 9 (cont'd)

Table 8. Data and Empirical Survivor Function for Failure Mode 1 $S_n(t)$

Year	Time to Failure (Years)	Number of Occurrences of Failure Mode 1 (Strength)	Number of Occurrences of Failure Mode 2 (Fatigue)	Survivor Function for Failure Mode1 (Strength)
1999	15	2	44	0.998241
2000	16	1	55	0.998190
2001	17	2	64	0.998087
2002	18	1	73	0.998036
2003	19	1	67	0.997984

Empirical Reliability Analysis Using Life Data

- Example 9 (cont'd)
  - The complete data set covers years from 1984 till 2060.
  - The results are plotted in Figure 9 as a step function.
  - The figure also shows the fitted reliability function using loglinear transformation and regression as discussed in Example 11.

Empirical Reliability Analysis Using Life Data

- Example 9 (cont'd)

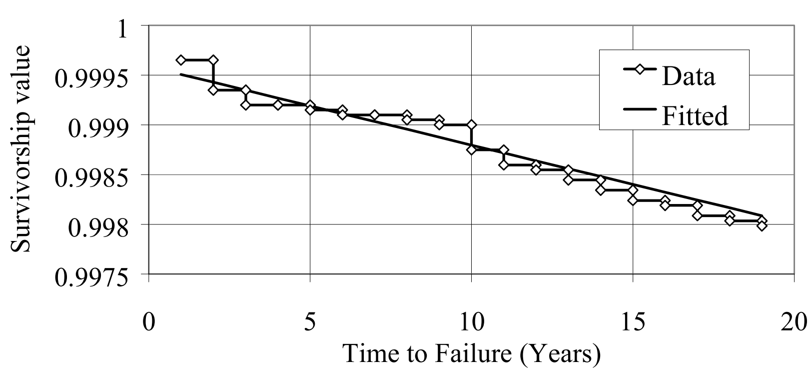


Figure 9. Empirical Survivor Function for Example 9

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parametric Reliability Functions
    - Besides the traditional distribution estimation methods, such as the method of moments and maximum likelihood described in Appendix A, the empirical survivor functions can be used to fit analytical reliability functions.
    - After evaluating an empirical reliability function, an analytical parametric hazard rate functions, such as given by Eqs. 45 and 47, can be fitted using the empirical survivorship function obtained from life data.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parametric Reliability Functions (cont'd)
    - The Weibull reliability function was used in studies performed for the U. S. Army Corps of Engineers as provided in Eq. 39 including the exponential reliability function as its specific case.
    - Also, the reliability function having a polynomial cumulative hazard function was used as follows:

$$R(t) = \exp(-H(t)) \tag{60a}$$

$$H(t) = a_0 + a_1\, t + a_2\, t_2 \tag{60b}$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parametric Reliability Functions (cont'd)
    - Therefore, the hazard function is given by

$$h(t) = a_1 + 2\, a_2\, t \qquad (60c)$$

    - For the special case where the parameters a_0 and a_2 are equal to zero, Eq. 60b reduces to the exponential distribution.
    - For the special case where the parameters a_0 and a_1 are zeros, the Eq. 60b reduces to the specific case of the Weibull distribution with the shape parameter of 2 (Rayleigh distribution)

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parameter Estimation Using Loglinear Transformation
    - Eqs. 60a to 60c provide exponential models with parameters a_0, a_1, and a_2. The logarithmic transformation of Eqs. 60a to 60c leads to

$$-\ln(R(t)) = a_0 + a_1\, t \qquad (61a)$$

$$-\ln(R(t)) = a_0 + a_1\, t + a_2\, t^2 \qquad (61b)$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parameter Estimation Using Loglinear Transformation
    - Using y to denote the left side of these equation, i.e., $y = -\ln(R(t))$, the following solutions can be obtained for the parameters a's according to Eq. 61a:

$$a_1 = \frac{\sum t_i y_i - \frac{1}{n}\sum t_i \sum y_i}{\sum t_i^2 - \frac{1}{n}\left(\sum t_i\right)^2} \qquad a_0 = \frac{\sum y_i}{n} - \frac{a_1 \sum x_i}{n}$$

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Parameter Estimation Using Loglinear Transformation
    - The parameters of Eq. 61b can obtained by solving the following set of simultaneous equations:

$$na_0 + a_1 \sum t_i + a_2 \sum t_i^2 = \sum y_i$$
$$a_0 \sum t_i + a_1 \sum t_i^2 + a_2 \sum t_i^3 = \sum t_i y_i$$
$$a_0 \sum t_i^2 + a_1 \sum t_i^3 + a_2 \sum t_i^4 = \sum t_i^2 y_i$$

Empirical Reliability Analysis Using Life Data

- Example 10: Loglinear Transformation for Parameter Estimation for Example 7 Data
  - For Example 7 data, the loglinear least square estimation gives the following values of the parameter estimates:

$$a_0 = 0.263018$$

$$a_1 = -\ 0.013930 \quad (1/\text{Year})$$

$$a_2 = 0.000185 \quad (1/\text{Year2})$$

Empirical Reliability Analysis Using Life Data

- Example 10: (cont'd)
  - All the model parameters estimates are of high statistical significance.
  - The multiple adjusted correlation coefficient squared (R^2) is 0.999 indicating a good fit.
  - The fitted values of reliability function and the respective empirical survivor function are given in Table 9 and Figure 8.
  - Example 4.11 of your Textbook shows another example of estimating the parameters.

Empirical Reliability Analysis Using Life Data

Table 9. Empirical Survivor Function, $S_n(t)$, and Fitted Reliability Function Using Loglinear Transformation and Regression for Example 10

Year	Time to Failure (Years)	Number of Failures	Survivor Function	Fitted Reliability Function
1937	0	0	1.000000	–
⋮	⋮	⋮	⋮	⋮
1973	36	0	1.000000	–
⋮	⋮	⋮	⋮	⋮
2002	65	172	0.868300	0.870060
2003	66	177	0.859450	0.861140
2004	67	179	0.850500	0.851996

Empirical Reliability Analysis Using Life Data

- Example 10 (cont'd)

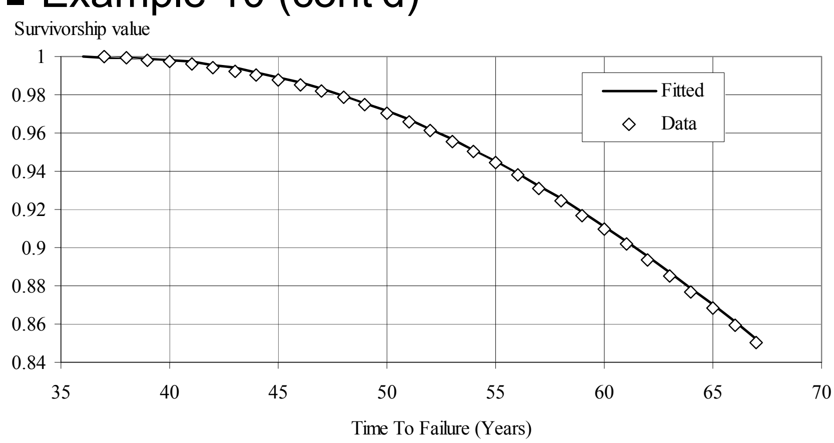


Figure 8. Fitted Reliability Functions using Lolinear Transformation and Regression for Example 10

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Nonlinear Model Estimation
    - The model provided by Eqs. 60a and 60b is nonlinear with respect to time with three parameters.
    - The parameters can be estimated, and errors can be analyzed using nonlinear regression analysis procedures.
    - The estimation of nonlinear model parameters can be essentially based on using numerical optimization methods.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Nonlinear Model Estimation (cont'd)
    - The parameter estimates can be obtained using the quasi Newton method of optimization.
    - A numerical algorithm can be advised for this purpose, or commercially available software, such as STATISTICA and its *Nonlinear Estimation* procedure, can be used.
    - **Example 4.12** and **13** of your Textbook illustrate the nonlinear estimation procedures.

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Probability Plotting
    - Probability plots are visual representations that show reliability data and preliminary estimation of assumed TTF distribution parameters, by graphing transformed values of an empirical survivor function (or CDF) versus time (or transformed time) on a specially constructed probability paper.
    - Reliability data that follow the underlying distribution of a probability paper type will fall on a straight line.

Empirical Reliability Analysis Using Life Data

- Example 14: Probability Plotting of Weibull Distribution for the Data of Example 8
  - A transformation of the reliability Weibull function can be developed by taking the logarithm of the reliability function of Eq. 39 twice as follows:

$$\ln \ln\left(\frac{1}{R(t)}\right) = \beta \ln t - \beta \ln \alpha \qquad (62)$$

Empirical Reliability Analysis Using Life Data

- Example 14 (cont'd)
  - Let

$$y = \ln\left[\ln\left(\frac{1}{R(t)}\right)\right] \quad \text{and} \quad x = \ln(t)$$

  - y is therefore linear in x with slope β.
  - Replacing $R(t)$ by the respective empirical survivor function, i.e., $S_n(t)$, a linear regression procedure can be used to fit the following line to the transformed data:

Empirical Reliability Analysis Using Life Data

- Example 14 (cont'd)

$$y(x) = bx + a$$

  - The distribution parameters can be estimated as follows:

$$\beta = b \text{ and } \alpha = \exp(-a/\beta)$$

  - The values of these estimates for the data of Example 8 are

$$\beta = 0.5554$$
$$\alpha = 1543246.1$$
$$a = -7.91411$$

Empirical Reliability Analysis Using Life Data

- Example 14 (cont'd)
  - The fitted reliability function and the respective empirical survivor function are given in Table 13.
  - The respective probability plot is given in Figure 10.
  - The sum of the squared residuals for the Weibull distribution fitted using the probability paper is 0.000000271, which is worse compared to 0.000000128 based on the nonlinear estimation in Example 4-13 of the textbook.

Empirical Reliability Analysis Using Life Data

Table 13. Empirical Survivor Function, $S_n(t)$, and Fitted Weibull Reliability Function Using Probability Paper for Example 14

Year	Time to Failure (Years)	Number of Occurrences of Failure Mode 1 (Strength)	Survivor Function for Failure Mode1 (Strength)	Probability Paper Fitted Reliability Function
1984	0	0	1.000000	-
⋮	⋮	⋮	⋮	⋮
1998	14	2	0.998343	0.998418
1999	15	2	0.998241	0.998356
2000	16	1	0.998190	0.998297
2001	17	2	0.998087	0.998238
2002	18	1	0.998036	0.998181
2003	19	1	0.997984	0.998126

Empirical Reliability Analysis Using Life Data

- Example 14 (cont'd)

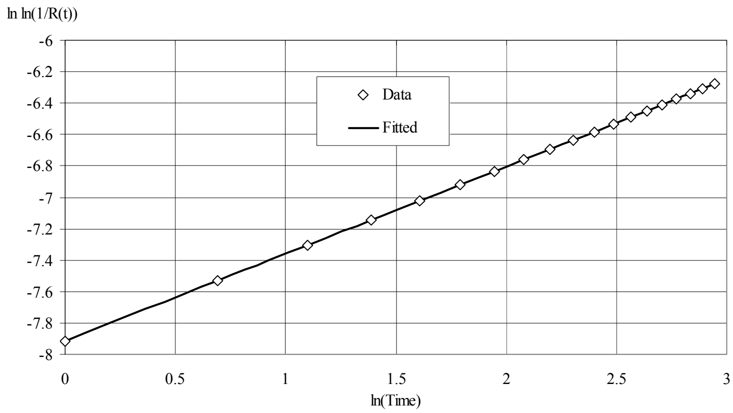


Figure 10. Weibull Probability Paper Plotting for Example 14

Empirical Reliability Analysis Using Life Data

- Selection and Fitting Reliability Models
  - Assessment of Hazard Functions
    - Once the parameters of underlying life distributions are known, i.e., estimated, assessing the cumulative hazard function (CHRF) and hazard (failure) rate function is reduced to applying Eq. 46 and 47, respectively.
    - Two examples of the hazard functions calculations are provided for demonstration purposes:
      - Reliability function with a polynomial CHRF (Eq. 60)
      - Based on the Weibull reliability function from Example 14

Empirical Reliability Analysis Using Life Data

- <u>Example 15:</u> Hazard Function Assessment from a Polynomial Cumulative Hazard Function
  - Example 4-12 demonstrated the development of a polynomial cumulative hazard function from reliability data.
  - The resulting reliability function expressed according to Eq. 60 with the estimated parameters is as follows:

$$R(t) = \exp(-0.262649 + 0.013915t - 0.000185t^2)$$

Empirical Reliability Analysis Using Life Data

- Example 15 (cont'd)
  - Using Eq. 46, the CHRF is

$$H(t) = 0.262649 - 0.013915t + 0.000185t^2$$

where t is time in years.

  - The respective hazard (failure) rate function is the derivative of $H(t)$, as provided by Eq. 47, therefore it can be written as

$$h(t) = -0.013915 + 0.000370t$$

Empirical Reliability Analysis Using Life Data

- Example 15 (cont'd)
  - The results of these calculations are given in Table 14 and Figure 11.
  - Taking into account that the hazard rate functions are used for projections, the table covers years from 1990 till 2010.
  - It can be observed from the figure that the hazard (failure) rate function is increasing in time, which shows aging of the given equipment.

Empirical Reliability Analysis Using Life Data

- Example 15 (cont'd)

Table 14. Hazard (Failure) Rate and Cumulative Hazard Rate Functions for Reliability Function with a Polynomial CHRF for Example 4-12 Data and Example 15 Computations

Year	Time to Failure (Years)	Hazard Rate Function	Cumulative Hazard Rate Function
1980	43	0.001995	0.006369
⋮	⋮	⋮	⋮
2007	70	0.011985	0.195099
2008	71	0.012355	0.207269
2009	72	0.012725	0.219809
2010	73	0.013095	0.232719

Empirical Reliability Analysis Using Life Data

- Example 15 (cont'd)

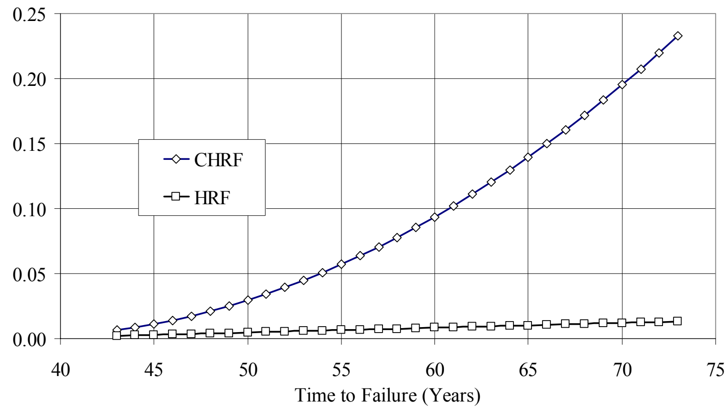


Figure 11. Cumulative Hazard Rate Function (CHRF) and Hazard Rate Function (HRF) for Example 15

Empirical Reliability Analysis Using Life Data

- <u>Example 16:</u> Assessing the Hazard Function for the Weibull Distribution
  - This example is based on the Weibull reliability function obtained using probability plotting in Example 14.
  - The Weibull CHRF $H(t)$ is given by Eq. 52 and the respective hazard (failure) rate function $h(t)$ by Eq. 51.

Empirical Reliability Analysis Using Life Data

- Example 16 (cont'd): Assessing the Hazard Function for the Weibull Distribution
  - Using these equations and the estimates of the distribution parameters from Example 14, the following expressions for $H(t)$ and $h(t)$ can be obtained:

$$H(t) = (t/1543246.1)^{0.5554}$$

$$h(t) = (0.5554/1543246.1)(t/1543246.1)^{0.5554-1}$$

$$= 3.60\text{x}10^{-7}(t/1543246.1)^{-0.4446}$$

Empirical Reliability Analysis Using Life Data

- Example 16 (cont'd)
  - The resulting hazard functions are given in Table 15.
  - Contrary to the previous example, the hazard (failure) rate function in this case is decreasing in time, which shows that the given unit is improving with respect to failure mode 1 which ***might not be realistic***.
  - If it is not realistic, a *different probability* distribution should be considered.

Empirical Reliability Analysis Using Life Data

- Example 16 (cont'd)

Table 15. Hazard (Failure) Rate and Cumulative Hazard Rate Functions for Weibull Reliability Function for Example 14 Data and Example 16 Computations

Year	Time to Failure (Years)	Hazard Rate Function	Cumulative Hazard Rate Function
1985	1	0.000203025	0.000366
⋮	⋮	⋮	⋮
2008	24	4.94205E-05	0.002136
2009	25	4.85316E-05	0.002185
2010	26	4.76927E-05	0.002233

Bayesian Methods

- The procedures discussed in the previous sections are related to the so-called statistical inference.
- Applying any of such procedures is usually associated with some assumptions, e.g., a sample is composed of *uncorrelated identically distributed random variables*.
- The "identically distributed" property can be stated according to a specific distribution, e.g., the exponential or Weibull distribution.

Bayesian Methods

- Such an assumption sometimes is checked using appropriate hypothesis testing procedures.
- Nevertheless, even if the corresponding hypothesis is not rejected, these characteristics cannot be taken with absolute certainty.
- In the framework of statistics, data result from observations, tests, measurements, polls, etc. These data can be viewed as *objective* information.

Bayesian Methods

- Types of Information

The types of information available to engineers can be classified as:

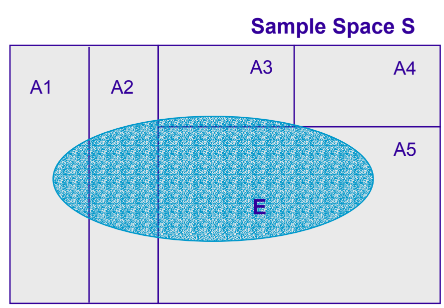
1. **Objective** information based on experimental results, or observations; and
2. **Subjective** information based on experience, intuition, other previous problems that are similar to the one under consideration, or the physics of the problem.

Bayesian Methods

- Bayesian Probabilities
  - Problems with both <u>objective</u> and <u>subjective</u> types of information.
  - The subjective probabilities are assumed to constitute a prior knowledge about a parameter, with gained objective information (or probabilities).
  - Combining the two type produces posterior knowledge.
  - The combination is performed based on Bayes' theorem.

Bayesian Methods

- Bayes' Theorem



$$P(A_i|E)=\frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1)+P(A_2)P(E|A_2)+...+P(A_n)P(E|A_n)}$$

Bayesian Methods

- Example (cont'd): Defective Products
  - Consider Line 3 of the three manufacturing lines.
  - The three lines manufacture 20%, 30%, and 50% of the components, respectively.
  - The quality assurance department of the producing factory determined that the probability of having defective products from lines 1, 2, and 3 are 0.1, 0.1, and 0.2, respectively.

Bayesian Methods

- Example (cont'd): Defective Products
  - The following events were defined:

L_1	=	Component produced by line 1
L_2	=	Component produced by line 2
L_3	=	Component produced by line 3
D	=	Defective component

  - Therefore, the following probabilities are given:

$P(D\|L_1)=$	0.1
$P(D\|L_2) =$	0.1
$P(D\|L_3) =$	0.2

Bayesian Methods

- Example (cont'd): Defective Products
  - Since these events are not independent, the joint probabilities can be determined as follows:

$$P(D \cap L_1) = P(D|L_1)P(L_1) = 0.1(0.2) = 0.02$$

$$P(D \cap L_2) = P(D|L_2)P(L_2) = 0.1(0.3) = 0.03$$

$$P(D \cap L_3) = P(D|L_3)P(L_3) = 0.2(0.5) = 0.1$$

Bayesian Methods

- Example (cont'd): Defective Products
  - The theorem of total probability can be used to determine the probability of a defective component as follows:

$$\begin{aligned} P(D) &= P(D|L_1)\,P(L_1) + P(D|L_2) + P(D|L_3)\,P(L_3) \\ &= 0.1(0.2) + 0.1(0.3) + 0.2(0.5) = 0.02 + 0.03 + 0.1 \\ &= 0.15 \end{aligned}$$

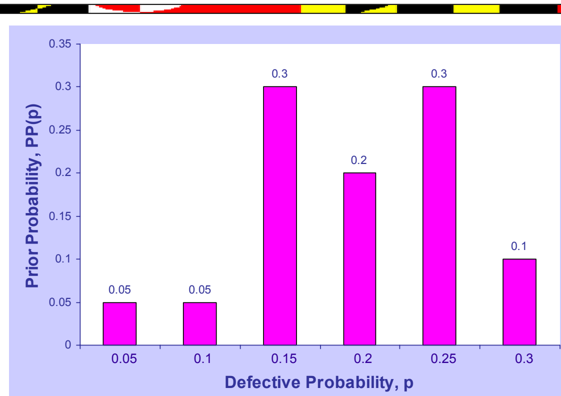
  - Therefore, on the average, 15% of the components produced by the factory are defective.

Bayesian Methods

- Example (cont'd): Defective Products
  - Because of the high contribution of Line 3 to the defective probability, a quality assurance engineer subjected the line to further analysis.
  - The defective probability for Line 3 was assumed to be 0.2. An examination of the source of this probability revealed that it is subjective, and also is uncertain.
  - A better description of this probability can be as shown in a figure in the form of a prior discrete distribution for the probability. The distribution is called $P_P(p)$.

Bayesian Methods

Example (cont'd):
Defective Products



The mean probability $\bar{p}$ based on this distribution is:

$\bar{p}$ = 0.05(0.05)+0.1(0.05)+0.15(0.3)+0.2(0.2)+0.25(0.3)+0.3(0.1)
0.1975

which is approximately 0.2.

Bayesian Methods

- Example (cont'd): Defective Products
  - Now assume that a component from Line 3 was tested and found to be defective, the subjective prior distribution needs to be revised to reflect the new (objective) information.
  - The revised distribution is called the posterior distribution ($P'_P(p)$), and can be computed as follows:

$$P(A_i|E)=\frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1)+P(A_2)P(E|A_2)+...+P(A_n)P(E|A_n)}$$

Bayesian Methods

- Example (cont'd): Defective Products

$$P'_P(0.05)=\frac{0.05(0.05)}{0.1975}=0.012658$$

$$P'_P(0.1)=\frac{0.05(0.1)}{0.1975}=0.0253165$$

$$P(A_i|E)=\frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1)+P(A_2)P(E|A_2)+...+P(A_n)P(E|A_n)}$$

$$P'_P(0.15)=\frac{0.15(0.3)}{0.1975}=0.2278481$$

$$P'_P(0.2)=\frac{0.2(0.2)}{0.1975}=0.2025316$$

$$P'_P(0.25)=\frac{0.25(0.3)}{0.1975}=0.03797468$$

$$P'_P(0.3)=\frac{0.3(0.1)}{0.1975}=0.1518987$$

Bayesian Methods

- Example (cont'd): Defective Products
  - The resulting probabilities add up to 1. The mean probability p based on the posterior distribution is:

$$\begin{aligned}\overline{P} &= 0.05(0.012658) + 0.1(0.025316) \\ &\quad + 0.15(0.227848) + 0.2(0.202532) \\ &\quad + 0.25(0.379747) + 0.3(0.151899) \\ &= 0.218354\end{aligned}$$

Bayesian Methods

- Example (cont'd): Defective Products
  - The posterior mean probability (0.218354) is larger than the prior mean probability (0.1975). The increase is due to the detected failure from the test.
  - Now assume that a second component from Line 2 was tested and found to be defective, the posterior distribution needs to be revised to reflect the new (objective) information. The revised posterior distribution builds on the current posterior distribution, treating it as a prior distribution.

Bayesian Methods

- Example (cont'd): Defective Products

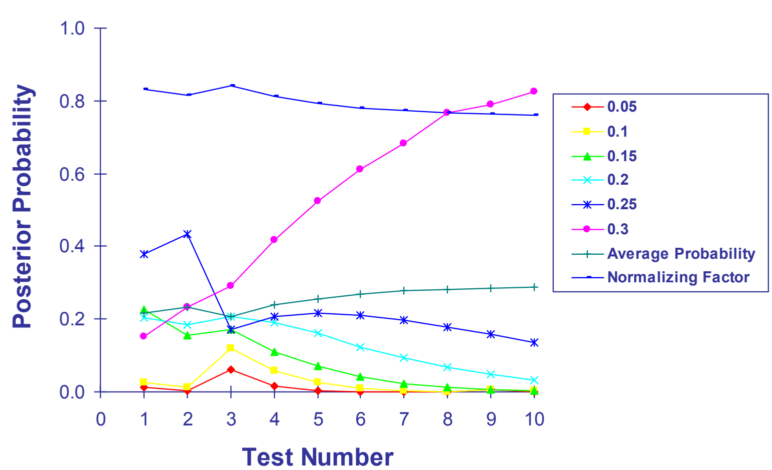
Probability	Prior	Post. 1D	Post. 2D	Post. 3D	Post. 4D	Post. 5D	Post. 6D	Post. 7D	Post. 8D	Post. 9D	Post. 10D
0.05	0.05	0.012658	0.002899	0.061036	0.014622	0.003064	0.00596	0.000111	2E-05	3.55E-06	6.2E-07
0.1	0.05	0.025316	0.011594	0.121008	0.057978	0.024302	0.009456	0.003518	0.001271	0.00045	0.000157
0.15	0.3	0.227848	0.156522	0.154897	0.111324	0.069992	0.040852	0.022796	0.012356	0.006565	0.003439
0.2	0.2	0.202532	0.185507	0.199432	0.191108	0.160205	0.124675	0.092763	0.067037	0.047492	0.033174
0.25	0.3	0.379747	0.434783	0.172995	0.207218	0.217138	0.211225	0.19645	0.177461	0.157151	0.137217
0.3	0.1	0.151899	0.208696	0.290632	0.417751	0.525299	0.613196	0.684362	0.741855	0.788339	0.826012
Average, $\bar{p}$	0.1975	0.218354	0.233188	0.208712	0.238579	0.256997	0.268803	0.27675	0.282311	0.286318	0.289274
Normalizing Factor, ND	0.8525	0.831646	0.816812	0.841288	0.811421	0.793003	0.781197	0.77325	0.767689	0.763682	0.750726

The last row of the table is the normalizing factor for cases where a non-defective component results from a test. The factor in this case is denoted ND in the table. For example, the normalizing factor (ND) in case of a non-defective test according to the prior distribution is:

ND = 0.05(1-0.05) + 0.1(1-0.05) + 0.15(1-0.3) + … + (0.3)(1-0.1) = 0.8525

Bayesian Methods

- Example (cont'd): Defective Products



Bayesian Methods

- Example (cont'd): Defective Products
  - If the next 8 tests result in one nondefective and seven defective components, the resulting posterior distributions are shown in the table.
  - It can be observed from the figure that the average probability is approaching 0.3 as more and more defective tests are obtained.
  - The average probability cannot exceed 0.3 because the prior distribution has zero probability values for p values larger than 0.3.
  - Also, the effect of a non-defective component on the posterior probabilities can be seen in this figure.