CHAPTER

4a

RELIABILITY ASSESSMENT

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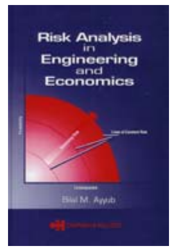

Risk Analysis for Engineering

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Introduction

"The reliability of an engineering system can be defined as its ability to fulfill its design purpose defined as performance requirements for some time period and environmental conditions. The theory of probability provides the fundamental bases to measure this ability."

Introduction

- The reliability assessment methods can be based on
  1. Analytical strength-and-load performance functions, or
  2. Empirical life data.
- They can also be used to compute the reliability for a given set of conditions that are time invariant or for a time-dependent reliability.

Introduction

- The reliability of a component or system can be assessed in the form of a probability of meeting satisfactory performance requirements according to some performance functions under specific service and extreme conditions within a stated time period.
- Random variables with mean values, variances, and probability distribution functions are used to compute probabilities.

Analytical Performance-Based Reliability Assessment

- First-Order Second Moment (FOSM) Method.
- Advanced Second Moment Method
- Computer-Based Monte Carlo Simulation

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method

$$Z = Z(X_1, X_2, \ldots, X_n) = \text{Supply - Demand} \qquad (1a)$$

$$Z = Z(X_1, X_2, \ldots, X_n) = \text{Structural strength - Load effect} \qquad (1b)$$

$$Z = Z(X_1, X_2, \ldots, X_n) = R\text{-}L \qquad (1c)$$

Z = performance function of interest
R = the resistance or strength or supply
L = the load or demand as illustrated in Figure 1

Analytical Performance-Based Reliability Assessment

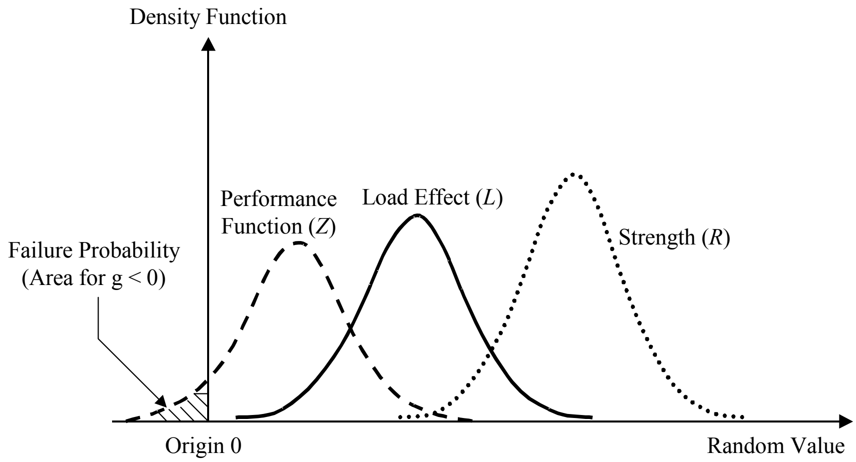


Figure 1. Performance Function for Reliability Assessment

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - The failure surface (or the limit state) of interest can be defined as $Z = 0$.
  - When $Z < 0$, the element is in the failure state, and when $Z > 0$ it is in the survival state.
  - If the joint probability density function for the basic random variables X_i's is $f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n)$, then the failure probability P_f of the element can be given by the integral

$$P_f = \int \cdots \int f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n)dx_1 dx_2 \ldots dx_n \qquad (2)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Where the integration is performed over the region in which $Z < 0$.
  - In general, the joint probability density function is unknown, and the integral is a formidable task.
  - For practical purposes, alternate methods of evaluating Pf are necessary. Reliability is assessed as one minus the failure probability.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Reliability Index
    - Instead of using direct integration (Eq. 2), performance function Z in Eq. 1 can be expanded using Taylor series about the mean value of *Xs* and then truncated at the linear terms. Therefore, the first-order approximation for the mean and variance are as follows:

$$\mu_Z \approx Z(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}) \qquad (3)$$

$$\sigma_Z^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial Z}{\partial X_i}\right)\left(\frac{\partial Z}{\partial X_j}\right)Cov(X_i, X_j) \qquad (4a)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Reliability Index (cont'd)

Where

μ = mean of random variable

μ_Z = mean of Z

σ_Z^2 = variance of Z

$Cov(X_i, X_j)$ = covariance of X_1 and X_2

$\frac{\partial Z}{\partial X_i}$ = partial derivative evaluated at the mean of random variable

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Reliability Index (cont'd)
    - For uncorrelated random variables, the variance cab be expressed as

$$\sigma_Z^2 \approx \sum_{i=1}^{n} \sigma_{X_i}^2 \left(\frac{\partial Z}{\partial X_i}\right)^2 \qquad (4b)$$

- The reliability index β can be computed from:

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_L}{\sqrt{\mu_R^2 + \mu_L^2}} \qquad (5)$$

$$P_f = 1 - \Phi(\beta) \qquad (6)$$

If z is assumed normally distributed.

Analytical Performance-Based Reliability Assessment

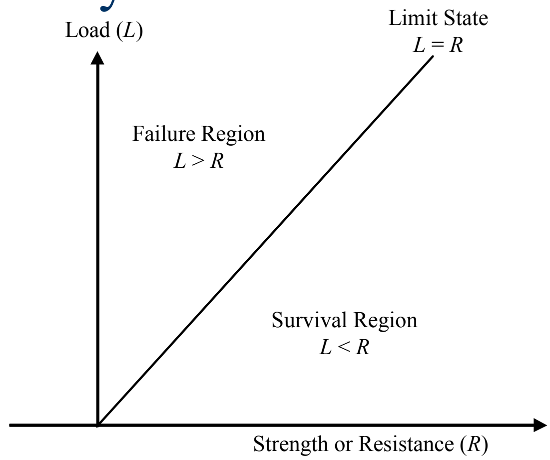


Figure 2. Performance Function for a Linear, Two-Random Variable Case

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions
    - For nonlinear performance functions, the Taylor series expansion of Z in linearized at some point on the failure surface referred to as the ***design point*** or ***checking point*** or ***the most likely failure point*** rather than at the mean.
    - Assuming X_i variables are uncorrelated, the following transformation to reduced or normalized coordinates can be used:

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \qquad (8a)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont'd)
    - It can be shown that the reliability index β is the shortest distance to the failure surface from the origin in the reduced *Y*-coordinate system.
    - The shortest distance is shown in Figure 3, and the reduced coordinates are

$$Y_L = \frac{\sigma_R}{\sigma_L} Y_R + \frac{\mu_R - \mu_L}{\sigma_L} \qquad (8b)$$

Analytical Performance-Based Reliability Assessment

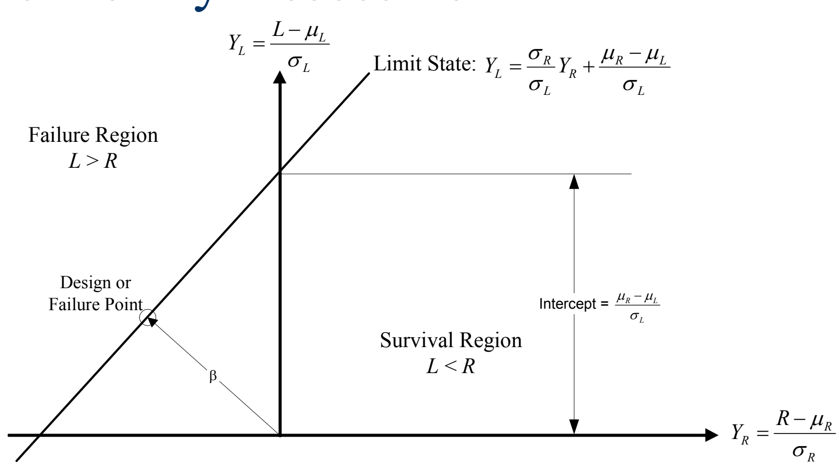


Figure 3. Performance Function for a Linear, Two-Random Variable Case in Normalized Coordinates

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont'd)
    - The concept of the shortest distance applies for a nonlinear performance function, as shown in Figure 4.
    - The reliability index β and the design point,

$$(X_1^*, X_2^*, \ldots, X_n^*)$$

can be determined by solving the following system of nonlinear equations iteratively for β:

Analytical Performance-Based Reliability Assessment

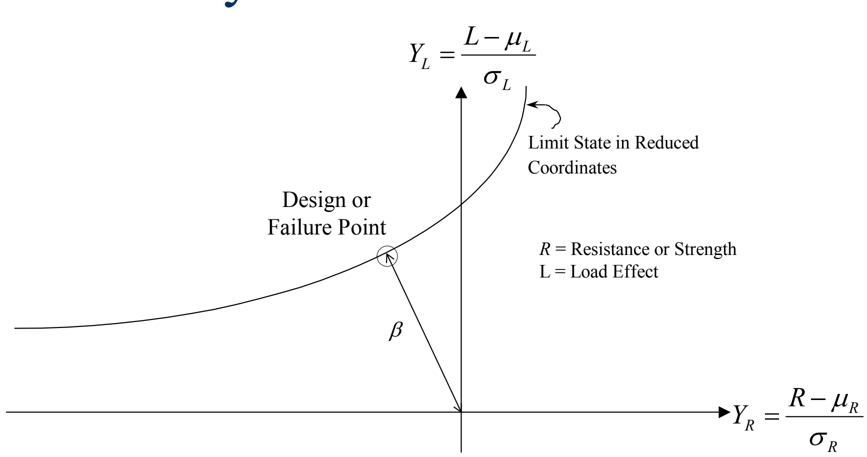


Figure 4. Performance Function for a Nonlinear, Two-Random Variable Case in Normalized Coordinates

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont'd)

$$\alpha_i = \frac{\left(\frac{\partial Z}{\partial X_i}\right)\sigma_{X_i}}{\left[\sum_{i=1}^{n}\left(\frac{\partial Z}{\partial X_i}\right)^2 \sigma_{X_i}^2\right]^{1/2}} \quad (9)$$

$$X_i^* = \mu_{X_i} - \alpha_i \beta \sigma_{X_i} \quad (10)$$

$$Z(X_1^*, X_2^*, \ldots, X_n^*) = 0 \quad (11)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont'd)
    - Where α_i is the directional cosine, and the partial derivatives are evaluated at the design point.
    - Eq. 6 can be used to compute P_f.
    - However, the above formulation is limited to normally distributed random variables.
    - The directional cosines are considered as measure of the importance of the corresponding random variables in determining the reliability index β.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Nonlinear Performance Functions (cont'd)
    - Also, partial safety factors γ that are used in load and resistance factor design (LRFD) can be calculated from

$$\gamma = \frac{X^*}{\mu_X} \qquad (12)$$

    - Generally, partial safety factors take on values larger than 1 loads, and less than 1 for strengths.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Equivalent Normal Distributions
    - If a random variable X is not normally distributed, then it must be transformed to an equivalent normally distributed random variable.
    - The parameters of the equivalent normal distribution are

$$\mu_{X_i}^N \quad \text{and} \quad \sigma_{X_i}^N$$

    - These parameters can be estimated by imposing two conditions.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Equivalent Normal Distributions (cont'd)

    First condition can be expressed as

$$\Phi\left(\frac{X_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N}\right) = F_i(X_i^*) \qquad (13a)$$

Second condition can be expressed as

$$\phi\left(\frac{X_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N}\right) = f_i(X_i^*) \qquad (13b)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Equivalent Normal Distributions (cont'd)

    where

F_i = non-normal cumulative distribution function

f_i = non-normal probability density function

Φ = cumulative distribution function of the standard normal variate

ϕ = probability density function of the standard normal variate.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Equivalent Normal Distributions (cont'd)
    - The standard deviation and mean of equivalent normal distributions are give by

$$\sigma_{X_i}^{N} = \frac{\phi(\Phi^{-1}[F_i(X_i^*)])}{f_i(X_i^*)} \qquad (14a)$$

$$\mu_{X_i}^{N} = X_i^* - \Phi^{-1}[F_i(X_i^*)]\sigma_{X_i}^{N} \qquad (14b)$$

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Equivalent Normal Distributions (cont'd)
    - Once $\sigma_{X_i}^{N}$ and $\mu_{X_i}^{N}$ are determined for each random variable, β can be solved following the same procedure of Eqs. 9 through 11.
    - The advanced second moment (ASM) method can deal with
      - Nonlinear performance function, and
      - Non-normal probability distributions

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Correlated Random Variables
    - A correlated (and normal) pair of random variables X_1 and X_2 with a correlation coefficient ρ can be transformed into noncorrelated pair Y_1 and Y_2 by solving for two eigenvalues and the corresponding eigenvectors as follows:

$$Y_1 = \frac{1}{2t}\left(\frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} + \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}}\right) \qquad (15a)$$

Analytical Performance-Based Reliability Assessment

$$Y_2 = \frac{1}{2t}\left(\frac{X_1 - \mu_{X_1}}{\sigma_{X_1}} - \frac{X_2 - \mu_{X_2}}{\sigma_{X_2}}\right) \qquad (15b)$$

- where $t = \sqrt{0.5}$. The resulting Y variables are noncorrelated with respective variances that are equal to the eigenvalues (λ) as follows:

$$\sigma_{Y_1}^2 = \lambda_1 = 1 + \rho \qquad (16a)$$

$$\sigma_{Y_2}^2 = \lambda_2 = 1 - \rho \qquad (16b)$$

Analytical Performance-Based Reliability Assessment

- For a correlated pair of random variables, Eqs. 9 and 10, have to be revised, respectively, to

$$\alpha_{Y_1} = \frac{\left[\left(\frac{\partial Z}{\partial X_1}\right) t\sigma_{X_1} + \left(\frac{\partial Z}{\partial X_2}\right) t\sigma_{X_2}\right]\sqrt{1+\rho}}{\left[\left(\frac{\partial Z}{\partial X_1}\right)^2 \sigma_{X_1}^2 + \left(\frac{\partial Z}{\partial X_2}\right)^2 \sigma_{X_2}^2 + 2\rho\left(\frac{\partial Z}{\partial X_1}\right)\left(\frac{\partial Z}{\partial X_2}\right)\sigma_{X_1}\sigma_{X_2}\right]^{1/2}} \quad (17a)$$

$$\alpha_{Y_2} = \frac{\left[\left(\frac{\partial Z}{\partial X_1}\right) t\sigma_{X_1} - \left(\frac{\partial Z}{\partial X_2}\right) t\sigma_{X_2}\right]\sqrt{1-\rho}}{\left[\left(\frac{\partial Z}{\partial X_1}\right)^2 \sigma_{X_1}^2 + \left(\frac{\partial Z}{\partial X_2}\right)^2 \sigma_{X_2}^2 + 2\rho\left(\frac{\partial Z}{\partial X_1}\right)\left(\frac{\partial Z}{\partial X_2}\right)\sigma_{X_1}\sigma_{X_2}\right]^{1/2}} \quad (17b)$$

Analytical Performance-Based Reliability Assessment

and

$$X_1^* = \mu_{X_1} - \sigma_{X_1} t\beta\left(\alpha_{Y_1}\sqrt{\lambda_1} + \alpha_{Y_2}\sqrt{\lambda_2}\right) \quad (18a)$$

$$X_2^* = \mu_{X_2} - \sigma_{X_2} t\beta\left(\alpha_{Y_1}\sqrt{\lambda_1} - \alpha_{Y_2}\sqrt{\lambda_2}\right) \quad (18b)$$

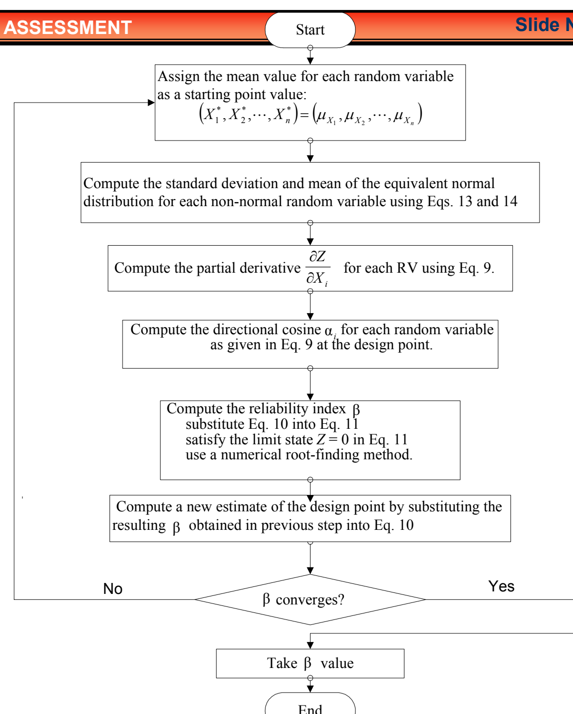
where the partial derivatives are evaluated at the design point.

Analytical Performance-Based Reliability Assessment

- Advanced Second-Moment Method
  - Numerical Algorithms
    - The advanced second moment (ASM) method can be used to assess the reliability of a structure according to nonlinear performance function that may include non-normal random variables.
    - Implementation of the method require efficient and accurate numerical algorithms.
    - The ASM algorithms are provided in the following two flowcharts for
      - **Noncorrelated random variables** (**Case a**)
      - **Correlated random variables** (**Case b**)

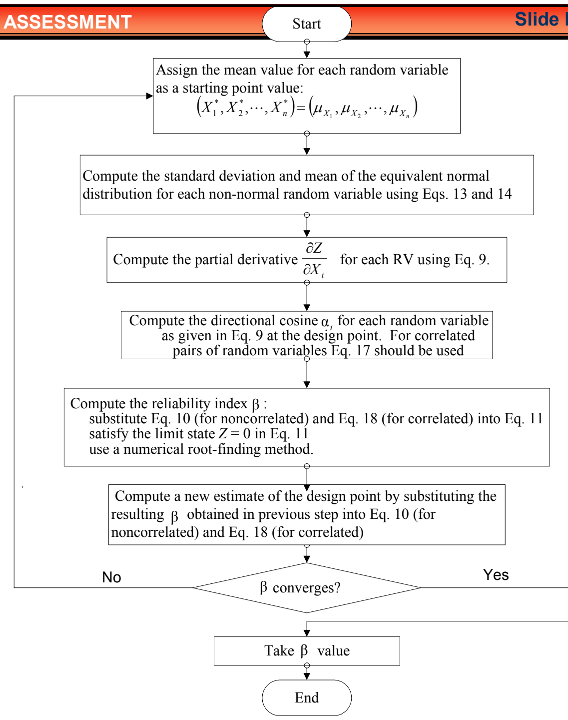
Case a:

Non-correlated Random Variables



Case b:

Correlated Random Variables



Analytical Performance-Based Reliability Assessment

- Example 1: Reliability Assessment Using a Nonlinear Performance Function
  - The strength-load performance function for a components is assumed to have the following form:

$$Z = X_1 X_2 - \sqrt{X_3}$$

where X's are basic random variables with the following probabilistic characteristics:

Random Variable	Mean Value (μ)	Standard Deviation (σ)	Coefficient of Variation	Case (a) Distribution Type	Case (b) Distribution Type
X_1	1	0.25	0.25	Normal	Lognormal
X_2	5	0.25	0.05	Normal	Lognormal
X_3	4	0.80	0.20	Normal	Lognormal

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Using first-order reliability analysis based on first-order Taylor series, the following can be obtained from Eqs. 3 to 5:

$$\mu_Z \cong (1)(5) - \sqrt{4} = 5 - 2 = 3$$

$$\sigma_Z \cong \sqrt{(5)^2(0.25)^2 + (1)^2(0.25)^2 + (-0.5/\sqrt{4})^2(0.8)^2}$$
$$= \sqrt{1.5625 + 0.0625 + 0.04} = 1.2903$$

$$\beta \cong \frac{\mu_Z}{\sigma_Z} = \frac{3}{1.2903} = 2.325$$

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - These values are applicable to both cases (a) and (b). Using advanced second-moment reliability analysis, the following table can be constructed for cases (a) and (b):

Case (a): Iteration 1

Random Variable	Failure Point	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}$	Directional Cosines (α)
X_1	1.000E+00	1.250E+00	9.687E-01
X_2	5.000E+00	2.500E-01	1.937E-01
X_3	4.000E+00	-2.000E-01	-1.550E-01

Analytical Performance-Based Reliability Assessment

Case (a): Iteration 1

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - The derivatives in the above table are evaluated at the failure point. The failure point in the first iteration is assumed to be the mean values of the random variables.
  - The reliability index can be determined by solving for the root according to Eq. 11 for the limit state of this example using the following equation:

$$Z = \left(\mu_{X_1} - \alpha_1 \beta \sigma_{X_1}\right)\left(\mu_{X_2} - \alpha_2 \beta \sigma_{X_2}\right) - \sqrt{\mu_{X_3} - \alpha_3 \beta \sigma_{X_3}} = 0$$

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 2.37735 for this iteration.

Case (a): Iteration 2

Random Variable	Failure Point	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}$	Directional Cosines (a)
X_1	4.242E-01	1.221E+00	9.841E-01
X_2	4.885E+00	1.061E-01	8.547E-02
X_3	4.295E+00	-1.930E-01	-1.555E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 2.3628 for this iteration.

Case (a): Iteration 3

Random Variable	Failure Point	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}$	Directional Cosines (α)
X_1	4.187E-01	1.237E+00	9.846E-01
X_2	4.950E+00	1.047E-01	8.329E-02
X_3	4.294E+00	-1.930E-01	-1.536E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 2.3628 for this iteration which means that β has converged to 2.3628.
  - The failure probability =1-Φ(β) = 0.009068.
  - The partial safety factors can be computed as:

Random Variable	Failure Point	Partial Safety Factors
X_1	0.418378	0.418378
X_2	4.950849	0.99017
X_3	4.290389	1.072597

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Case (b)
    - The parameters of the lognormal distribution can be computed for three random variables based on their respective means (μ) and deviations (σ) as follows:

$$\sigma_Y^2 = \ln\left[1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right] \quad \text{and} \quad \mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2$$

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
    - The results of these computations are summarized as follows:

Random Variable	Distribution Type	First Parameter (μ_Y)	Second Parameter (σ_Y)
X_1	Lognormal	-0.03031231	0.24622068
X_2	Lognormal	1.608189472	0.04996879
X_3	Lognormal	1.366684005	0.20

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

Case (b): Iteration 1

Equivalent Normal					
Random Variable	Failure Point	Standard Deviation	Mean Value	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}^N$	Directional Cosines (α)
X_1	1.000E+00	2.462E-01	9.697E-01	1.231E+00	9.681E-01
X_2	5.000E+00	2.498E-01	4.994E+00	2.498E-01	1.965E-01
X_3	4.000E+00	7.922E-01	3.922E+00	-1.980E-01	-1.557E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function

  - The derivatives in the above table are evaluated at the failure point. The failure point in the first iteration is assumed to be the mean values of the random variables.
  - The reliability index can be determined by solving for the root according to Eq. 11 for the limit state of this example using the following equation:

$$Z = \left(\mu_{X_1}^N - \alpha_1 \beta \sigma_{X_1}^N\right)\left(\mu_{X_2}^N - \alpha_2 \beta \sigma_{X_2}^N\right) - \sqrt{\mu_{X_3}^N - \alpha_3 \beta \sigma_{X_3}^N} = 0$$

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 2.30530 for this iteration.

Case (b): Iteration 2

	Equivalent Normal				
Random Variable	Failure Point	Standard Deviation	Mean Value	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}^N$	Directional Cosines (α)
X_1	4.202E-01	1.035E-01	7.718E-01	5.050E-01	9.118E-01
X_2	4.881E+00	2.439E-01	4.992E+00	1.025E-01	1.850E-01
X_3	4.206E+00	8.330E-01	3.912E+00	-2.031E-01	-3.667E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 3.3224 for this iteration.

Case (b): Iteration 3

	Equivalent Normal				
Random Variable	Failure Point	Standard Deviation	Mean Value	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}^N$	Directional Cosines (α)
X_1	4.584E-01	1.129E-01	8.020E-01	5.465E-01	9.118E-01
X_2	4.843E+00	2.420E-01	4.991E+00	1.109E-01	1.850E-01
X_3	4.927E+00	9.758E-01	3.803E+00	-2.198E-01	-3.667E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 3.3126 for this iteration.

Case (b): Iteration 4

	Equivalent Normal				
Random Variable	Failure Point	Standard Deviation	Mean Value	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}^N$	Directional Cosines (α)
X_1	4.612E-01	1.136E-01	8.041E-01	5.499E-01	9.118E-01
X_2	4.843E+00	2.420E-01	4.991E+00	1.116E-01	1.850E-01
X_3	4.989E+00	9.880E-01	3.789E+00	-2.212E-01	-3.667E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 3.3125 for this iteration.

Case (b): Iteration 5

	Equivalent Normal				
Random Variable	Failure Point	Standard Deviation	Mean Value	$\frac{\partial Z}{\partial X_i}\sigma_{X_i}^N$	Directional Cosines (α)
X_1	4.612E-01	1.136E-01	8.041E-01	5.500E-01	9.118E-01
X_2	4.843E+00	2.420E-01	4.991E+00	1.116E-01	1.850E-01
X_3	4.989E+00	9.880E-01	3.789E+00	-2.212E-01	-3.667E-01

Analytical Performance-Based Reliability Assessment

- Example 1 (cont'd): Reliability Assessment Using a Nonlinear Performance Function
  - Therefore, β = 3.3125 for this iteration which means that β has converged to 3.3125.
  - The failure probability =1-$\Phi(\beta)$ = 0.0004619.
  - The partial safety factors can be computed as:

Random Variable	Failure Point	Partial Safety Factors
X_1	0.461189	0.461189
X_2	4.843135	0.968627
X_3	4.988968	1.247242

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Monte Carlo simulation (MCS) techniques are basically sampling processes that are used to estimate the failure probability of a component or system.
  - The basic random variables in Eq. 1, that is

$$Z = Z(X_1, X_2, \ldots, X_n) = R\text{-}L$$

are randomly generated and substituted into above equation.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Then the fraction of the cases that resulted in failure are determined to assess the failure probability.
  - Three methods are described herein:
    1. Direct Monte Carlo Simulation
    2. Conditional Expectation
    3. The Importance Sampling Reduction Method

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Direct Monte Carlo Simulation Method
    - In this method, samples of the basic noncorrelated variables are drawn according to their corresponding probabilities characteristics and fed into performance function Z as given by Eq. 1.
    - Assuming that N_f is the number of simulation cycles for which $Z < 0$ in N simulation cycles, then an estimate of the mean failure probability can be expressed as

$$\overline{P}_f = \frac{N_f}{N} \quad (19)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Direct Monte Carlo Simulation Method (cont'd)
    - The variance of the estimated failure probability can be approximately computed using the variance expression for a binomial distribution as:

$$Var(\overline{P}_f) = \frac{(1-\overline{P}_f)\overline{P}_f}{N} \qquad (20)$$

    - Therefore, the coefficient of variation (COV) of the estimated failure probability is

$$\text{COV}(\overline{P}_f) = \frac{1}{\overline{P}_f}\sqrt{\frac{(1-\overline{P}_f)\overline{P}_f}{N}} \qquad (21)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Direct Monte Carlo Simulation Method (cont'd)
    - Some of the advantages of this method is that it is easy to implement and understand.
    - The disadvantages include:
      - Expensive in some cases, especially if the failure probabilities are small.
      - Inefficient
    - The importance sampling method (IS) is described later for the purpose of increasing the efficiency of the is method.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation
    - This method can also be used to estimate the failure probability according to the performance function of Eq. 1.
    - The method requires generating all the basic random variables in Eq. 1 except the random variables with the highest variability (i.e., COV), which is used as a control variable, X_k.
    - The conditional expectation is computed as the cumulative distribution function.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation (cont'd)
    - For the following performance function:

$$Z = R - L \qquad (22)$$

and for a randomly generated value of L or R, the failure probability for each cycle is given, respectively, as

$$P_{f_i} = F_R(l_i) \qquad (23)$$

$$P_{f_i} = 1 - F_L(r_i) \qquad (24)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation (cont'd)
    - In these equations, L and R are the control variables. The total failure probability P_f can be estimated from

$$\overline{P}_f = \frac{\sum_{i=1}^{N} P_{f_i}}{N} \qquad (25)$$

    - Where N is the number of simulation cycles.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Conditional Expectation (cont'd)
    - The accuracy of Eq. 25 can be estimated through the variance and coefficient of variation as given by

$$Var(\overline{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \overline{P}_f)^2}{N(N-1)} \qquad (26)$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f} \qquad (27)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont'd)
    - To improve the efficiency of simulation when estimating the probability of failure for a given performance function, Importance Sampling (IS) techniques are used.
    - In some performance function, if the margin of safety Z is large and its variance is too small, larger simulation effort will be required to obtain sufficient simulation runs with satisfactory performances, i.e.,

smaller failure probabilities require larger number of simulation cycles

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont'd)
    - In this method, the basic random variables are generated according to some carefully selected probability distributions, i.e.,

$$\text{Importance density function, } h_X(\underline{x})$$

With mean values that are closer to the design point than their original (actual) probability distributions.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont'd)
    - The fundamental equation for this method is given by

$$\overline{P}_f = \frac{1}{N}\sum_{i=1}^{N} I_i \frac{f_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})}{h_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})} \quad (28)$$

N = number of simulation cycles
$f_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})$ = original joint density function of the basic random variables evaluated at the i^{th} generated values of the basic random variables
$h_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})$ = selected joint density function of the basic random variables evaluated at the i^{th} generated values of the basic random variables
I = performance indicator function that takes values of either 0 for failure and 1 for survival

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Importance Sampling (cont'd)
    - The coefficient of variation of the estimate failure probability can be based on the variance of a sample mean as follows:

$$COV(\overline{P}_f) = \frac{\sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}\left(I_i \frac{f_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})}{h_{\underline{X}}(x_{1i}, x_{2i}, \ldots, x_{ni})} - \overline{P}_f\right)^2}}{\overline{P}_f} \quad (29)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables
    - A correlated (and normal) pair of random variables X_1 and X_2 with a correlation coefficient ρ can transformed using linear regression transformation as follows:

$$X_2 = b_0 + b_1 X_1 + \varepsilon \qquad (30a)$$

b_0 = intercept of a regression line between X_1 and X_2
b_1 = slope of the regression line
ε = random (standard) error with a mean of zero and a standard deviation as given in Eq. 30d).

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables (cont'd)
    - These regression model parameters can be determined in terms of the probabilistic characteristics of X_1 and X_2 as follows::

$$b_1 = \frac{\rho \sigma_{X_2}}{\sigma_{X_1}} \qquad (30b)$$

$$b_0 = \mu_{X_2} - b_1 \mu_{X_1} \qquad (30b)$$

$$\sigma_\varepsilon = \sigma_{X_2} \sqrt{1-\rho^2} \qquad (30d)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Correlated Random Variables (cont'd)

Procedure for a correlated pair of random variables:

1. Compute the intercept of a regression line between X_1 and X_2 (b_0), the slope of the regression line (b_1), and the standard deviation of the random (standard) error (ε) using Eqs. 30b to 30d.
2. Generate a random (standard) error using a zero mean and a standard deviation as given by Eq. 30d.

Analytical Performance-Based Reliability Assessment

Procedure for a correlated pair of random variables (cont'd):

3. Generate a random value for X_1 using its probabilistic characteristics (mean and variance).
4. Compute the corresponding value of X_2 as follows (based on Eq. 30a): $x_2 = b_0 + b_1 x_1 + \varepsilon$ where b_0 and b_1 are computed in step 1; ε is a generated random (standard) error from step 2; and x_1 is generated value from step 3.
5. Use the resulting random (but correlated) values of x_1 and x_2 in the simulation-based reliability assessment methods.

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Time-Dependent Reliability Analysis
    - Several methods for analytical time-dependent reliability assessment are available.
    - In these methods, significant structural loads as a sequence of pulses that can be described by a Poisson process with mean occurrence rate, λ, random intensity, S, and duration, τ.
    - The limit state of the structure at any time can be defined as

$$R(t) - S(t) < 0 \qquad (31)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Time-Dependent Reliability Analysis (cont'd)
    - where $R(t)$ is the strength of the structure at time t and $S(t)$ is the loads at time t.
    - The instantaneous probability of failure can then be defined at time t as probability of $R(t)$ less than $S(t)$.
    - The reliability function, $L(t)$, was defined as the probability that the structure survives during interval of time $(0,t)$ as

$$L(t) = \int_0^\infty \exp[-\lambda t[1 - \frac{1}{t}\int_0^t Fs(g(t)r)dt] fR(r)dr \qquad (32a)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Time-Dependent Reliability Analysis (cont'd)
    - where $f_R(r)$ is the probability density function of an initial strength, R, and $g(t)$ is the time-dependent degradation in strength.
    - The reliability can be expressed in terms of the conditional failure rate or hazard function, $h(t)$ as

$$h(t) = -\frac{d}{dt}\ln L(t) \qquad (32b)$$

or

$$L(t) = \exp[-\int_0^t h(\xi)d\xi] \qquad (32c)$$

Analytical Performance-Based Reliability Assessment

- Monte Carlo Simulation Methods
  - Time-Dependent Reliability Analysis (cont'd)
    - The reliability $L(t)$ is based on the complete survival during the service life interval $(0,t)$.
    - It means the probability of successful performance during a service life interval $(0,t)$.
    - Therefore, the probability of failure, $P_f(t)$, can be computed as the probability of the complementary event, i.e., $P_f(t) = 1 - L(t)$ being not equivalent to $P[R(t) < S(t)]$.

Empirical Reliability Analysis Using Life Data

- Failure and Repair
  - The basic notion of reliability analysis based on life data is *time to failure*.
  - The useful life of a product can be measured in terms of its time to failure.
  - In addition to time, other possible exposure measures include the number of cycles to failure of mechanical, electrical, temperature or humidity.

Empirical Reliability Analysis Using Life Data

- Failure and Repair (cont'd)
  - If the failed product is subject to repair or replacement, it is called *repairable* (in opposite to *non-repairable* objects).
  - The respective repair or replacement requires some time to get done, which is called *time to repair/replace*
  - The time to failure is used for the non-repairable components or systems.

Empirical Reliability Analysis Using Life Data

- Failure and Repair (cont'd)
  - For repairable products, there is another important characteristic, which is called *time between failures*.
  - This is another random variable or a set of random variables.
  - It can be assumed that the time to the first failure is the same random variable as the time between the first and the second failures, the time between the second and the third failures, and so on.

Empirical Reliability Analysis Using Life Data

- Types of Data
  - Failure data often contain not only times to failure (the so-called distinct failures), but also times in use (or exposure length of time) that do not terminate with failures.
  - Such exposure time intervals terminating with non-failure are called *times to censoring* (TTC).
  - Therefore, life data of equipment can be classified into two types, *complete* and *censored* data

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - Failure data often contain not only times to failure (the so-called distinct failures), but also times in use (or exposure length of time) that do not terminate with failures.
  - Such exposure time intervals terminating with non-failure are called *times to censoring* (TTC).
  - Therefore, life data of equipment can be classified into two types, *complete* and *censored* data.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - The ***complete life data*** are commonly based on equipment tested to failure or times to failure based on equipment use, i.e., field data.
  - ***Censored life data*** include some observation results that represent only lower or upper limits on observation of times to failure.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - Censored data can be further classified into
    - Type I or
    - Type II
  - **Type I** data are based on observations of a life test, which for economical or other reasons, must be terminated at specified time t_0.
  - As the result, only the lifetimes of those units that have failed before t0 are known exactly.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - If, during the time interval $(0, t_0]$, s out of n sample units failed, then the information in the data set obtained consists of s observed, ordered times to failure as follows:

$$t_1 < t_2 < . . . < t_s \qquad (33a)$$

  - and the information that $(n - s)$ units have survived the time t_0.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - In some life data testing, testing is continued until a specified number of failures r is achieved, i.e., the respective test or observation is terminated at the r^{th} failure.
  - In this case, r is not random.
  - This type of testing, i.e., observation or field data collection, results in **Type II** censoring.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - It includes r observed ordered times to failure

$$t_1 < t_2 < . . . < t_r \qquad (33b)$$

  - And the information that $(n - r)$ units have survived the time t_r.
  - But, in opposite to Type I censoring, the test or observation duration t_r is random, which should be taken into account in the respective statistical estimation procedures.

Empirical Reliability Analysis Using Life Data

- Types of Data (cont'd)
  - In reliability engineering, Type I right-censored data are commonly encountered.
  - Figure 5 shows a summary of these data types.
  - Other types of data are possible such as random censoring.

Empirical Reliability Analysis Using Life Data

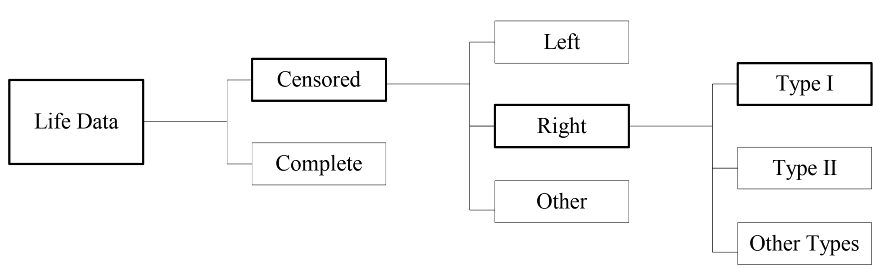


Figure 5. Types of Life Data

Empirical Reliability Analysis Using Life Data

- Example 2: Data of Distinct Failures
  - In this example, the following complete sample of 19 times to failure for a structural component given in years to failure is provided for illustration purposes:

26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 43, 50, 56

Empirical Reliability Analysis Using Life Data

- Example 3: Right Censored Data
  - In this example, tests of equipment are used for demonstration purposes to produce observations in the form of life data as given in Table 1.
  - The data in the table provide an example of Type I censored data (the sample size is 12), with time to censoring equal to 51 years.
  - If the data collection was assumed to terminate just after the 8th failure, the data would represent a sample of Type II right

Empirical Reliability Analysis Using Life Data

- Example 3 (cont'd): Right Censored Data

censored data with the same sample size of 12.

– The respective data are given in Table 2.

Table 1. Example of **Type I** Right Censored Data (in Years) for Equipment

Time Order Number	1	2	3	4	5	6	7	8	9	10	11	12
Time (Years)	7	14	15	18	31	37	40	46	51	51	51	51
TTF or TTC	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTC	TTC	TTC	TTC

TTF = time to failure, and TTC = time to censoring

Empirical Reliability Analysis Using Life Data

- Example 3 (cont'd): Right Censored Data

Table 2. Example of **Type II** Right Censored Data (in Years) for Equipment

Time Order Number	1	2	3	4	5	6	7	8	9	10	11	12
Time (Years)	7	14	15	18	31	37	40	46	46	46	46	46
TTF or TTC	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTF	TTC	TTC	TTC	TTC

TTF = time to failure, and TTC = time to censoring

Empirical Reliability Analysis Using Life Data

- Example 4: Random Censoring
  - Table 3 contains the time to failure data, in which two failure modes were observed.
  - The data in this example were generated using Monte Carlo simulation.
  - The simulation process is restarted once a failure occurs according to one of the modes at time *t*, making this time t for the other mode as a time to censoring.

Empirical Reliability Analysis Using Life Data

- Example 4 (cont'd): Random Censoring

Year	TTF (Years)	Number of Occurrences of a Given Failure Mode	
		Strength (FM1)	Fatigue (FM2)
1984	1	0	0
1985	2	7	0
1986	3	6	0
1987	4	3	0
1988	5	0	0
1989	6	1	7
1990	7	1	12
1991	8	0	20
1992	9	1	36
1993	10	1	47
1994	11	5	61
1995	12	3	33
1996	13	1	74
1997	14	2	65
1998	15	2	58
1999	16	2	44

Table 3.
Partial Data Set From 20,000Simulation Cycles for the Two Failure Modes of Strength and Fatigue for a Structural Component