CHAPTER

8a

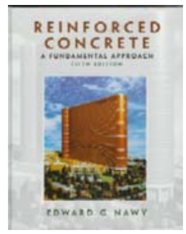
REINFORCED CONCRETE

A Fundamental Approach - Fifth Edition

SERVICEABILITY OF BEAMS AND ONE-WAY SLABS

A. J. Clark School of Engineering •Department of Civil and Environmental Engineering

SPRING 2004



By
Dr . Ibrahim. Assakkaf

ENCE 454 – Design of Concrete Structures

Department of Civil and Environmental Engineering

University of Maryland, College Park

Introduction

- Serviceability of a structure is determined by its
  - Deflection
  - Cracking
  - Extent of corrosion of its reinforcement
  - Surface deterioration of its concrete
- Surface deterioration can be minimized by proper control of mixing, placing, and curing of concrete.

Introduction

- This chapter deals with the evaluation of *deflection* and *cracking* behavior of beams and one-way slabs.
- It should give adequate background on the effect of cracking on the stiffness of the member, the short-and long-term deflection performance.
- Deflection of two-way slabs and plates is presented in Chapter 11.

Significant of Deflection

- Significant of Deflection Observation
  - The working stress (WSM) method of design limited the stress in concrete to about 45% of its compressive strength and the stress in the steel to less than 50% of its yield strength.
  - As a result, heavier sections with higher reserve strength resulted as compared to those obtained by the current ultimate strength design.

Significant of Deflection

- Significant of Deflection Observation
  - Higher strength concrete having values in excess of 12,000 psi (83 MPa) and higher-strength steels that are used in strength design have resulted in lower values of load factors (e.g., 1.4 to 1.2 for dead load) and reduced reserve strength.
  - Hence, more slender and efficient members are specified, with ***deflection becoming a more obvious controlling criteria***.

Significant of Deflection

- Significant of Deflection Observation
  - Excessive deflection of floor slab may cause dislocation in the partitions it supports.
  - Excessive deflection of a beam can damage a partition below.
  - Excessive deflection of a lintel beam above a window opening could crack the glass panels.
  - In the case of open floors and roofs such as top garage floors, ponding of water can result.
  - Therefore, deflection control criteria are needed.

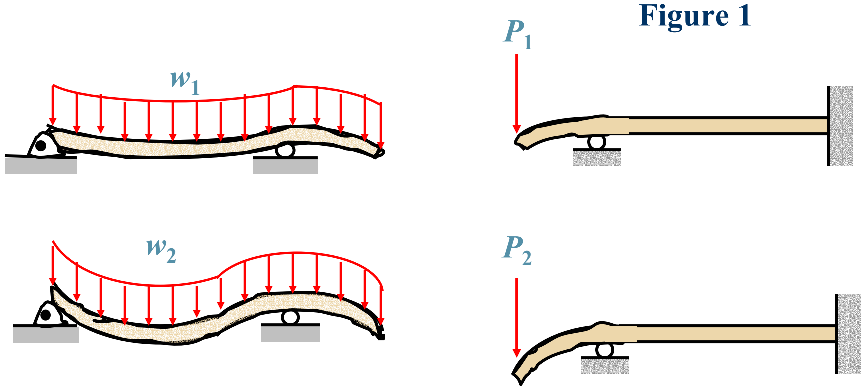
Significant of Deflection

- Theoretical Background
  - There are important relations between applied load and stress (flexural and shear) and the amount of deformation or deflection that a beam can exhibit.
  - In design of beams, it is important sometimes to limit the deflection for specified load.
  - So, in these situations, it is not enough only to design for the strength (flexural normal and shearing stresses), but also for excessive deflections of beams.

Significant of Deflection

- Theoretical Background (cont'd)

Figure 1

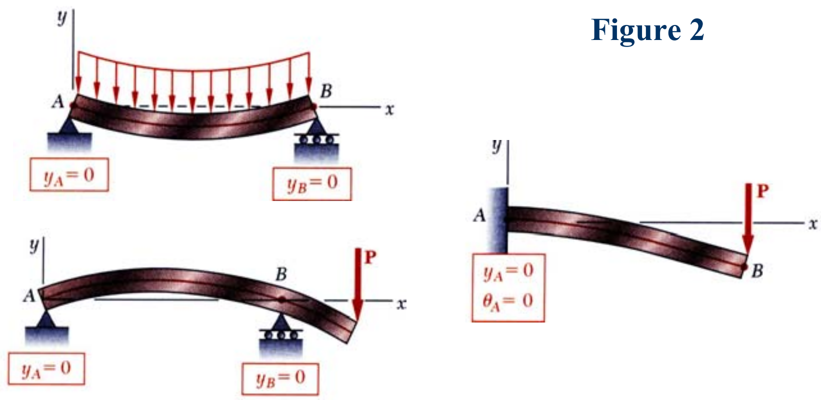


(a) $w_2 >> w_1$ (b) $P_2 >> P_1$

Significant of Deflection

- Theoretical Background (cont'd)

Figure 2



Significant of Deflection

- Theoretical Background (cont'd)
  - Figure 1 shows generally two examples of how the amount of deflections increase with the applied loads.
  - Failure to control beam deflections within proper limits in building construction is frequently reflected by the development of cracks in plastered walls and ceilings.

Significant of Deflection

- Theoretical Background (cont'd)
  - Beams in many structures and machines must deflect just right amount for gears or other parts to make proper contact.
  - In many instances the requirements for a beam involve:
    - A given load-carrying capacity, and
    - A specified maximum deflection

Significant of Deflection

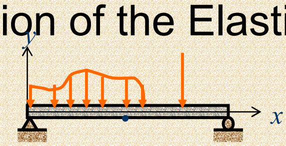
- Methods for Determining Beam Deflections
  - The deflection of a beam depends on four general factors:
    1. Stiffness of the materials that the beam is made of,
    2. Dimensions of the beam,
    3. Applied loads, and
    4. Supports

Significant of Deflection

- Methods for Determining Beam Deflections
  - Three methods are commonly used to find beam deflections:
    1) **The double integration method,**
    2) **The singularity function method, and**
    3) **The superposition method**

Beam Deflection

- The Differential Equation of the Elastic Curve for a Beam

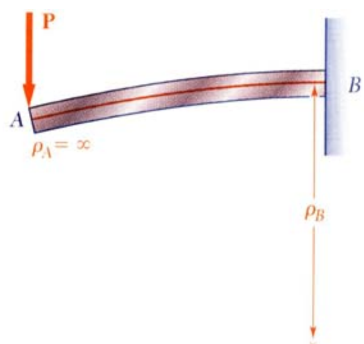


$$EI\frac{d^2y}{dx^2} = M(x) \qquad (1)$$

E = modulus of elsticity for the material
I = moment of inertia about the neutral axis of cross section
$M(x)$ = bending moment along the beam as a function of x

Beam Deflection

- The Differential Equation of the Elastic Curve for a Beam (cont'd)



- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

Beam Deflection

- The Differential Equation of the Elastic Curve for a Beam (cont'd)

- From elementary calculus, simplified for beam parameters,

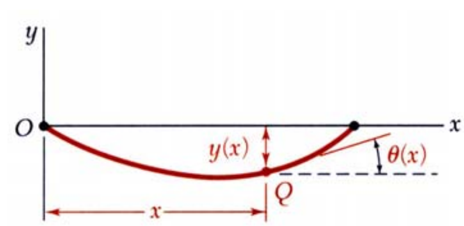
$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

- Substituting and integrating,

$$EI\frac{1}{\rho} = EI\frac{d^2y}{dx^2} = M(x)$$

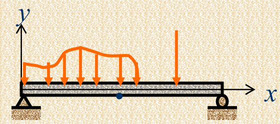
$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)dx + C_1x + C_2$$



Beam Deflection

- Sign Convention



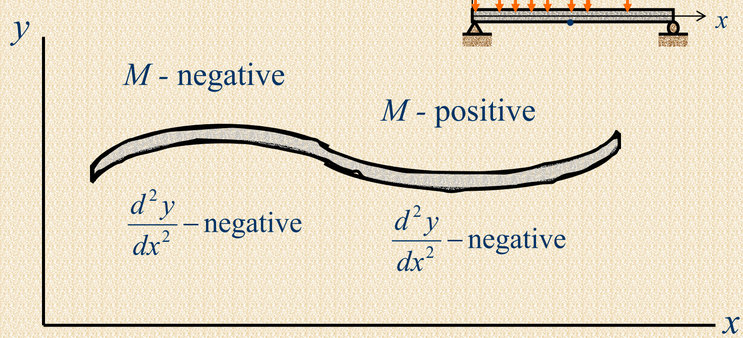
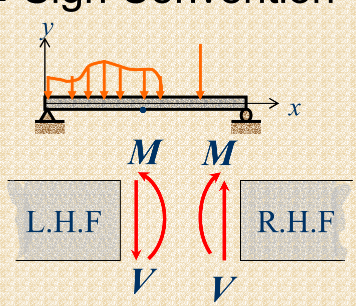


Figure 3. Elastic Curve

Beam Deflection

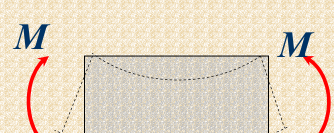
- Sign Convention

Figure 4



(a) Positive Shear & Moment

(b) Positive Shear (clockwise)

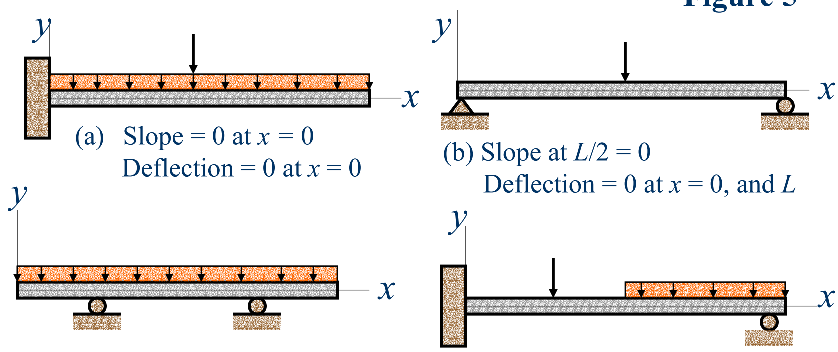


(c) Positive Moment (concave upward)

Beam Deflection by Integration

- Example Boundary Conditions

Figure 5



(a) Slope = 0 at x = 0
Deflection = 0 at x = 0

(b) Slope at $L/2$ = 0
Deflection = 0 at x = 0, and L

(c) Slope at rollers ?
Deflection at rollers = 0

(d) Slope = 0 at x =0
Deflection = 0 at x = 0 and $x = L$

Beam Deflection by Integration

- Example 1

A beam is loaded and supported as shown in the figure.

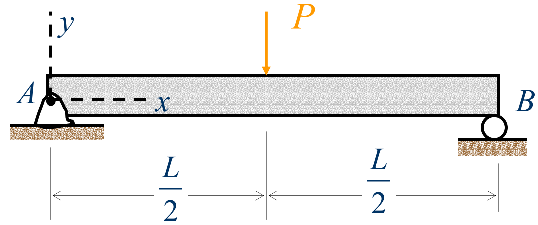
a) Derive the equation of the elastic curve in terms of P, L, x, E, and I.

b) Determine the slope at the left end of the beam.

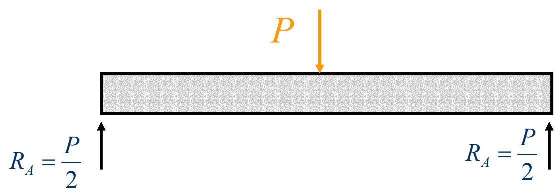
c) Determine the deflection at $x = L/2$.

Beam Deflection by Integration

- Example 1 (cont'd)

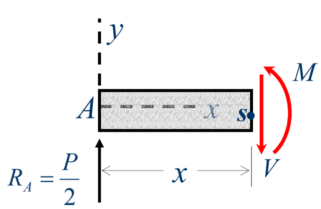


FBD



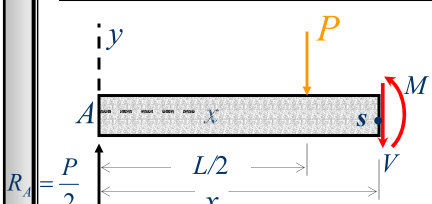
Beam Deflection by Integration

- Example 1 (cont'd)



$$+\circlearrowleft \sum M_S = 0; -M + \frac{P}{2}x = 0$$

$$\Rightarrow \boxed{M = \frac{P}{2}x \quad \text{for } 0 \le x \le L/2} \quad \text{(2a)}$$



$$\boxed{M = \frac{P}{2}x - P\left(x - \frac{L}{2}\right) \quad \text{for } L/2 \le x \le L}$$

(2b)

Beam Deflection by Integration

- Example 1 (cont'd)

Boundary conditions:

$\theta = 0$ at $x = L/2$ (from symmetry)

$y = 0$ at $x = 0$ and $x = L$

Using Eqs 1 and 2a:

$$EI\frac{d^2y}{dx^2} = M(x) = \frac{P}{2}x$$

$$EI\frac{dy}{dx} = EI\theta = \int M(x)dx = \int \frac{P}{2}x$$

$$EI\theta = \frac{P}{2}\frac{x^2}{2} + C_1 = \frac{P}{4}x^2 + C_1 \qquad (2c)$$

Beam Deflection by Integration

- Example 1 (cont'd)

Expression for the deflection y can be found by integrating Eq. 2c:

$$EIy = \int EI\theta dx = \int\left(\frac{P}{4}x^2 + C_1\right)$$

$$EIy = \frac{P}{4}\frac{x^3}{3} + C_1x + C_2$$

$$EIy = \frac{P}{12}x^3 + C_1x + C_2 \qquad (2d)$$

Beam Deflection by Integration

- Example 1 (cont'd)

The objective now is to evaluate the constants of integrations C_1 and C_2:

$$EI\theta(L/2) = 0$$

$$\frac{P}{4}x^2 + C_1 = 0 \Rightarrow \frac{P}{4}\left(\frac{L^2}{4}\right) + C_1 = 0$$

$$\boxed{C_1 = -\frac{PL^2}{16}} \qquad (2e)$$

Beam Deflection by Integration

- Example 1 (cont'd)

$$y(0) = 0$$

$$\frac{P}{12}x^3 + C_1x + C_2 = 0 \Rightarrow \frac{P}{12}(0) + C_1(0) + C_2 = 0$$

$$\boxed{C_2 = 0} \qquad (2f)$$

(a)The equation of elastic curve from Eq. 2d

$$EIy = \frac{P}{12}x^3 + C_1x + \cancel{C_2}^{\,0}$$

$$\boxed{y = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{PL^2}{16}x\right)} \qquad (2g)$$

Beam Deflection by Integration

- Example 1 (cont'd)

  (b) Slope at the left end of the beam:

  From Eq. 2c, the slope is given by

$$EI\theta = \frac{P}{4}x^2 + C_1 = \frac{P}{4}x^2 - \frac{PL^2}{16}$$

$$\theta = \frac{1}{EI}\left(\frac{P}{4}x^2 - \frac{PL^2}{16}\right)$$

  Therefore,

$$\theta_A = \theta(0) = \frac{1}{EI}\left(\frac{P}{4}(0)^2 - \frac{PL^2}{16}\right) = \boxed{-\frac{PL^2}{16}}$$

Beam Deflection by Integration

- Example 1 (cont'd)

  (c) Deflection at $x = L/2$:

  From Eq. 2g of the elastic curve:

$$y = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{PL^2}{16}x\right)$$

$$y(L/2) = \frac{1}{EI}\left(\frac{P}{12}\left(\frac{L}{2}\right)^3 - \frac{PL^2}{16}\left(\frac{L}{2}\right)\right)$$

$$= \boxed{-\frac{PL^3}{48EI}}$$

Beam Deflection by Superposition and Tables

- The slope or deflection of a beam is the sum of the slopes or deflections produced by the individual loads.
- The superposition method provides a means of quickly solving a wide range of more complicated problems by various combinations of known results.

Beam Deflection by Superposition and Tables

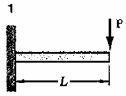
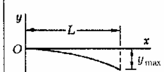
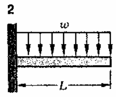
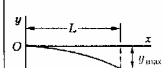
- To facilitate the task of designers or practicing engineers, most structural and mechanical handbooks include tables giving the deflections and slopes of beams for various loadings and types of support.
- Such a table can be found in your textbook (**Table 8.3**) and provided herein in the next few viewgraphs in Table 1.

Beam Deflection by Superposition and Tables

- Slopes and Deflection Tables

Table 1a

Appendix D. Beam Deflections and Slopes (Beer and Johnston 1992)

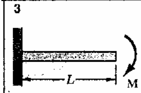
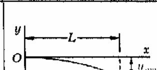
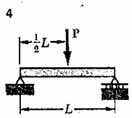
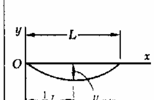
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$

Beam Deflection by Superposition and Tables

- Slopes and Deflection Tables

Table 1b

Appendix D. Beam Deflections and Slopes (Beer and Johnston 1992)

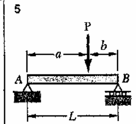
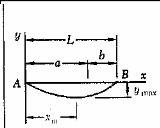
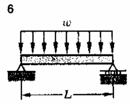
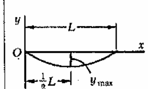
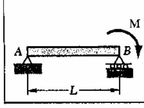
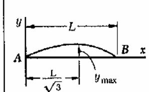
Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \leq \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$

Beam Deflection by Superposition and Tables

- Slopes and Deflection Tables

Table 1c

(Beer and Johnston 1992) Appendix D. Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
5 		For $a > b$: $-\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2-a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3-(L^2-b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4-2Lx^3+L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3-L^2x)$

Deflection Behavior of Beams

- The load-deflection relationship of a reinforced concrete beam is basically trilinear, as shown in Figure 6.
- It is composed of three regions prior to rupture:
  - – Region I
  - – Region II
  - – Region III

Deflection Behavior of Beams

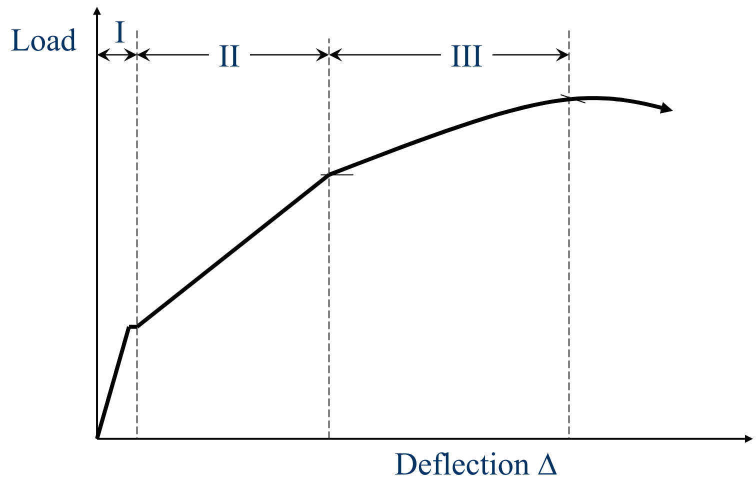


Figure 6. Beam-deflection relationship, Region I, precracking stage; region II, postcracking stage; region III, postserviceability stage (steel yields).

Deflection Behavior of Beams

- **Region I:** *precracking stage*, where a structural member is crack-free.
- **Region II**: *postcracking stage*, where the structural member develops acceptable controlled cracking both in distribution and width.
- **Region III**: *postserviceability* cracking stage, where the stress in the tension reinforcement reaches the limit state of yielding.

Deflection Behavior of Beams

- Precracking Stage: Region I
  - The maximum tensile stress is less than its tensile strength in flexure, that is less than the modulus of rupture f_r of concrete.
  - The flexural stiffness EI can be estimated using the Young's modulus of concrete and the uncracked reinforced concrete cross-section.
  - The load-deflection behavior depends on the stress-strain relationship of the concrete.

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - The value of E_c can be estimated using the ACI empirical expression as was discussed in Chapter 3:

$$E_c = 33 w_c^{1.5} \sqrt{f_c'} \qquad (3a)$$

or

$$E_c = 57{,}000 \sqrt{f_c'} \quad \text{for normal-weight concrete} \qquad (3b)$$

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - The precracking region stops at the initiation of the first flexural crack when the concrete stress reaches its modulus of rupture f_r.
  - The value of f_r for normal-weight concrete is

$$f_r = 7.5\sqrt{f'_c} \qquad (3c)$$

  - For all lightweight concrete, multiply the value of f_r by **0.75**
  - For sand-lightweight concrete, multiply the value of f_r by **0.85**

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:
    - Basically there are two approaches:
      1. **Gross section (simplified):** where I_g is calculated neglecting the presence of any reinforcing steel.
      2. **Transformed section (more accurate):** where I_{gt} is calculated taking into account the additional stiffness contributed by the steel reinforcement.

I_g = moment of inertia of the gross cross section
I_{gt} = moment of inertia of the gross cross section, including steel reinforcement stiffness

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:
    - The moment of inertia I_g for a rectangular section can be computed as

$$I_g = \frac{bh^3}{12} \qquad (4a)$$

    - And the corresponding cracking moment M_{cr} is

$$M_{cr} = \frac{I_g f_r}{y_t} = \frac{2 I_g f_r}{h} \qquad (4b)$$

Note that $y_t = h/2$ for rectangular section

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:

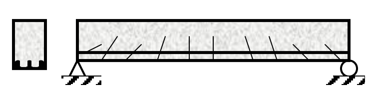
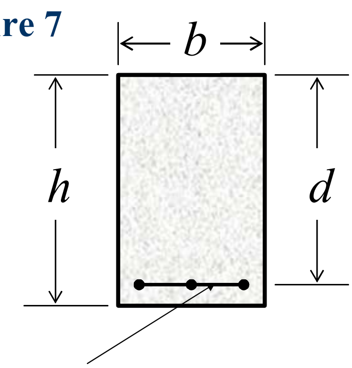


Figure 7



Neglect the stiffness provided by steel

$$I_g = \frac{bh^3}{12}$$

$$M_{cr} = \frac{I_g f_r}{y_t} = \frac{2 I_g f_r}{h}$$

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:
    - The calculation of the moment of inertia I_{gt} for a rectangular section in this case is dependent on the location of the neutral axis $\bar{y}$ of the transformed section as shown in Figure 8
    - The value of $\bar{y}$ for a rectangular section can be computed by

$$\bar{y} = \frac{(bh^2/2)(n-1)A_s d}{bh+(n-1)A_s} \qquad (5)$$

Note that $n = E_s/E_c$ = modular ratio

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:

$$\bar{y} = \frac{(bh^2/2)(n-1)A_s d}{bh+(n-1)A_s}$$

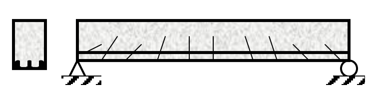
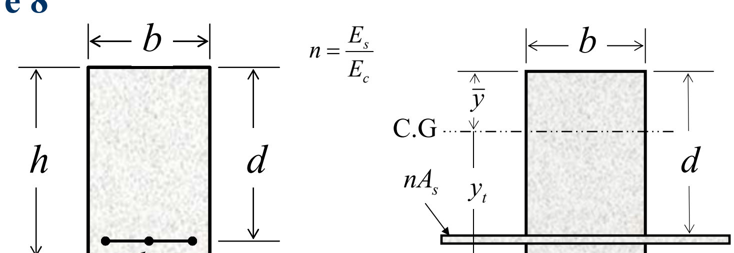


Figure 8



Do not neglect the stiffness provided by steel

Transformed Section

Deflection Behavior of Beams

- Precracking Stage: Region I (cont'd)
  - Estimation of the moment of inertia I in this region:
    - Once I_{gt} has been computed based on the location of the center of gravity, the cracking moment can be estimated from the flexure formula.
    - The corresponding cracking moment M_{cr} is

$$M_{cr} = \frac{I_{gt} f_r}{y_t} \qquad (6)$$

Deflection Behavior of Beams

- Postcracking Stage: Region II
  - The precracking region ends at the initiation of the first crack and moves into region II.
  - Most beams lie in this region at service loads.
  - Cracks are wider and deeper at midspan.
  - At limit state of service load cracking, the contribution of tension-zone concrete to flexural stiffness is neglected (see Figure 9).
  - The moment of inertia of the cracked section is designated I_{cr}.

Deflection Behavior of Beams

- Postcracking Stage: Region II

Stresses Elastic and Section Cracked

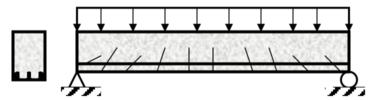
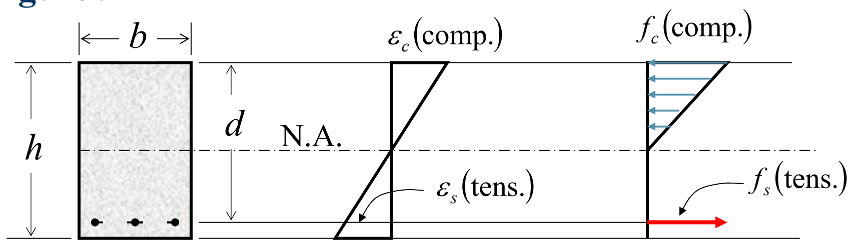


Figure 9



- Tensile stresses of concrete will be exceeded.
- Concrete will crack (hairline crack), and steel bars will resist tensile stresses.
- This will occur at approximately $0.5 f'_c$.

Deflection Behavior of Beams

- Postcracking Stage: Region II (cont'd)
  - Calculation of the moment of inertia I_{cr} in this region:
    - The moment of inertia I_{cr} for a rectangular section can be computed as

$$I_{cr} = \frac{bc^3}{3} + nA_s(d-c)^2 \qquad (7)$$

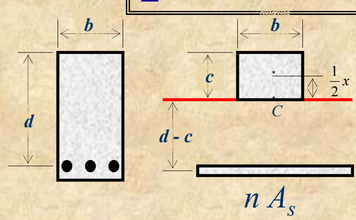
    - Where the value of the neutral axis c can be computed from the following quadratic equation:

$$\frac{1}{2}bc^2 + nA_s c - nA_s d = 0 \qquad (8)$$

Deflection Behavior of Beams

- Postcracking Stage: Region II (cont'd)
  - Calculation of the moment of inertia I_{cr} in this region:
    - The neutral axis for a concrete beam is found by solving the quadratic equation:

$$\frac{1}{2}bc^2 + nA_s c - nA_s d = 0 \qquad (8)$$



$$n = \frac{E_s}{E_c}$$

Deflection Behavior of Beams

- Postserviceability Cracking Stage and Limit State of Deflection Behavior at Failure: Region III
  - The load-deflection diagram is considerably flatter in this region due to substantial loss in stiffness of the section.
  - Extensive cracking takes place.
  - The beam is considered at this stage to have structurally failed by yielding of the tension steel, and continues to deflect without additional loading.

Deflection Behavior of Beams

- Postserviceability Cracking Stage and Limit State of Deflection Behavior at Failure: Region III
  - The cracks continue to open, and the neutral axis continues to rise upward until total crushing of the concrete when rupture occurs.
  - Postyield deflection and limit deflection at failure are not of major significant in design and hence are not being discussed in any detail, however, it is important to recognize the reserve deflection capacity as a measure of ductility in structures.

Deflection Behavior of Beams

- Concept of Effective Moment of Inertia I_e
  - In the case of Region II, in reality only of the beam cross-section is cracked.
  - As seen from Figure 9, the uncracked segments below the neutral axis along the beam span possess some degree of stiffness, which contributes to overall beam rigidity.
  - The actual stiffness lies between:

$$E_c I_g \text{ and } E_c I_{cr}$$

or

$$I_g \text{ and } I_{cr}$$

Deflection Behavior of Beams

- Concept of Effective Moment of Inertia I_e
  - Generally, as the load approaches the steel yield load level, the stiffness value approaches $E_c I_{cr}$.
  - The ACI Code recommends that deflection be calculated using an effective moment of inertia, I_e, where

$$I_g > I_e > I_{cr}$$

  - Once the effective moment of inertia is determined, the member deflection may be calculated by using standard deflection formulas.

Immediate Deflection

- Immediate deflection is the deflection that occurs as soon as load is applied on the member.
- The ACI Code, Section 9.5.2.3, states that this deflection may be calculated using a concrete modulus of elasticity E_c as given by Eq. 3, and an effective moment on inertia I_e as given by the following equations (Eqs. 9 and 10):

Immediate Deflection

- Effective Moment of Inertia

$$I_e = \left\{ \left(\frac{M_{cr}}{M_a} \right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a} \right)^3 \right] I_{cr} \right\} \leq I_g \quad (9)$$

ACI (9-8)

or

$$I_e = \left[I_{cr} + \left(\frac{M_{cr}}{M_a} \right)^3 \left(I_g - I_{cr} \right) \right] \leq I_g \quad (10)$$

Immediate Deflection

- Effective Moment of Inertia

I_e = effective moment of inertia
I_{cr} = moment of inertia of the cracked section transformed to concrete
I_g = moment of inertia of the gross (uncracked) concrete cross section about the centroidal axis, either neglecting or not neglecting steel
M_a = maximum moment in the member at the stage for which the deflection is being computed
M_{cr} = moment that would initially crack the cross section computed from

$$M_{cr} = \frac{f_r I_g}{y_t}$$

ACI (9-9)

Where
f_r = modulus of rupture for concrete as given by Eq. 3c
y_t = distance from the neutral axis of the uncracked cross section (neglecting steel) to the extreme tension fiber

Long-Term Deflection

- In addition to deflections that occur immediately, reinforced concrete members are subject to added deflections that occur gradually over long periods.
- These additional deflections are due mainly to creep and shrinkage and may eventually become excessive.
- The long-term deflections are computed based on:

Long-Term Deflection

1. the amount of sustained dead and live load, and,
2. The amount of compression reinforcement in the beam.

- The additional long-term deflections may be estimated as follows:

$$\Delta_{\mathrm{LT}} = \lambda \Delta_i = \left(\frac{T}{1 + 50\rho'} \right) \Delta_i \qquad (11)$$

Long-Term Deflection

Where

Δ_{LT} = additional long-term deflection
Δ_i = immediate deflection due to sustained loads
λ = a multiplier for additional long-term deflection [ACI Eq. (9-11)]
ρ' = nonprestressed compression reinforcement ratio (A'_s / bd)
T = time-dependent factor for sustained loads:

5 years or more	2.0
12 months	1.4
6 months	1.2
3 months	1.0

Long-Term Deflection

- Some judgment will be required in determining just what portion of the live loads should be considered *sustained*.
- In a residential applications, 20% sustained live load might be a logical estimate, whereas in storage facilities, 100% sustained load would be reasonable.