CHAPTER 8d. DIFFERENTIAL EQUATIONS

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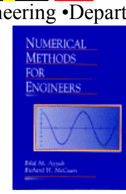

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

University of Maryland, College Park

First-order Ordinary Differential Equations

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- Runge-Kutta (RK) Methods
  - Euler's and modified Euler's methods are considered special cases of second-order Runge-Kutta (RK) method.
  - They are also considered to be one-step methods since they use information from one interval in estimating the value of y at the end of the interval.

First-order Ordinary Differential Equations

- Runge-Kutta (RK) Methods
  - A class of methods called Runge-Kutta methods, which include the Euler's and modified Euler's methods, is widely used.
  - In these methods, the y value at the end of the interval is determined based on the value at the beginning of the interval, the step size, and some representative slope over the interval.

First-order Ordinary Differential Equations

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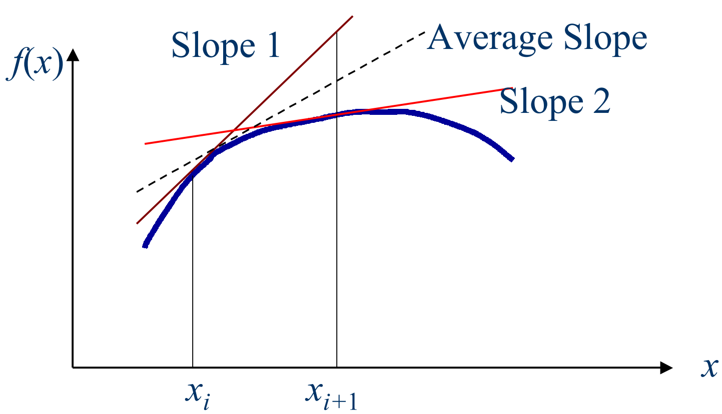
- Runge-Kutta (RK) Methods



First-order Ordinary Differential Equations

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- Second-order Runge-Kutta Methods
  - There are three representations for second-order RK Methods.
  - They are all similar in a sense that they use information at the beginning and at the end of the interval.
  - However, they differ in representing the slope, which projects the the new value of y at the end of the interval.

First-order Ordinary Differential Equations

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- First Representation
  - The first representation is the modified Euler's Method:

$$y_{i+1} = y_i + 0.5h\left[\underbrace{f(x_i, y_i)}_{\left.\frac{dy}{dx}\right|_{x_i}\ \text{Slope 1}\ S_1} + \underbrace{f(x_i + h, y_i + hf(x_i, y_i))}_{\left.\frac{dy}{dx}\right|_{x_{i+1}}\ \text{Slope 2}\ S_2}\right]$$

First-order Ordinary Differential Equations

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- First Representation of Second-order RK Methods

$$y_{i+1} = y_i + 0.5h[S_1 + S_2] \quad (19)$$

$$S_1 = f(x_i, y_i)$$
$$S_2 = f(x_i + h, y_i + hS_1)$$

First-order Ordinary Differential Equations

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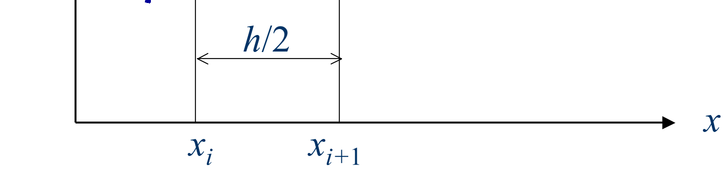
- First Representation of Second-order RK Methods



First-order Ordinary Differential Equations

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- Second Representation of Second-order RK Methods

$$y_{i+1} = y_i + S_2 h \qquad (20)$$

$$S_1 = f(x_i, y_i)$$
$$S_2 = f(x_i + 0.5h, y_i + 0.5hS_1)$$

First-order Ordinary Differential Equations

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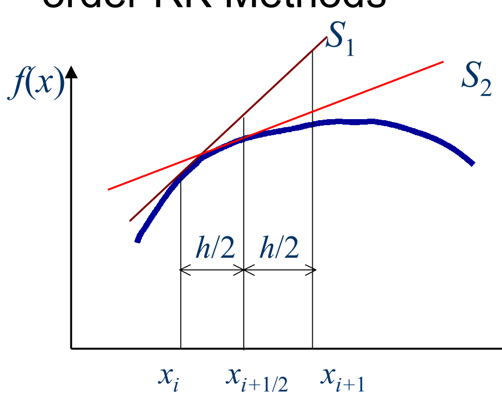
- Second Representation of Second-order RK Methods



$$S_1 = f(x_i, y_i)$$
$$S_2 = f(x_i + 0.5h, y_i + 0.5hS_1)$$

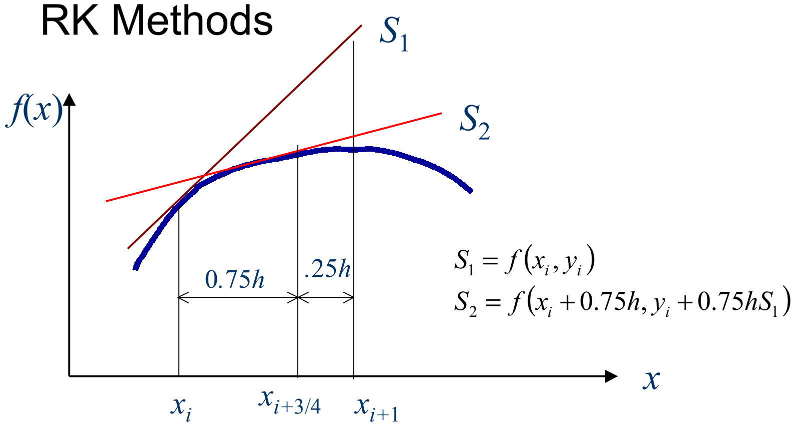
- Third Representation of Second-order RK Methods

$$y_{i+1} = y_i + \left(\frac{1}{3} S_1 + \frac{2}{3} S_2 \right) h \qquad (21)$$

$S_1 = f\left(x_i, y_i\right)$

$S_2 = f\left(x_i + 0.75h, y_i + 0.75hS_1\right)$

- Third Representation of Second-order RK Methods



First-order Ordinary Differential Equations

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- Example 10 – Third Representation Second-order RK Method

  Rework Example 9 using third representation second-order Runge-Kutta method:

$$\frac{dy}{dx} - \frac{1}{2} y = 0 \qquad \text{such that } y = 1 \text{ at } x = 0$$

First-order Ordinary Differential Equations

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- Example 10 (cont'd)

$$y_{i+1} = y_i + \left(\frac{1}{3} S_1 + \frac{2}{3} S_2 \right) h$$

$$S_1 = f(x_i, y_i) = \left. \frac{dy}{dx} \right|_{\substack{x=0 \\ y=1}} = \frac{1}{2} y = \frac{1}{2}(1) = \frac{1}{2}$$

$$S_2 = f(x_i + 0.75h, y_i + 0.75hS_1)$$

$$x_i + 0.75h = 0 + 0.75(0.1) = 0.075$$

First-order Ordinary Differential Equations

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- Example 10 (cont'd)

$$y_i + 0.75hS_1 = 1 + 0.75(0.1)\frac{1}{2} = 1.0375$$

$$S_2 = \left.\frac{dy}{dx}\right|_{\substack{x=0.075 \\ y=1.0375}} = \frac{1}{2}y = \frac{1}{2}(1.0375) = 0.51875$$

$$y_2 = y_1 + \left(\frac{1}{3}S_1 + \frac{2}{3}S_2\right)$$

$$= 1 + \left(\frac{1}{3}\frac{1}{2} + \frac{2}{3}(0.51875)\right)0.1 = 1.05125$$

First-order Ordinary Differential Equations

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- Third-order Runge-Kutta Methods

$$y_{i+1} = y_i + \frac{h}{6}\left[S_1 + 4S_2 + S_3\right] \quad (22)$$

$$S_1 = f(x_i, y_i)$$
$$S_2 = f(x_i + 0.5h, y_i + 0.5hS_1)$$
$$S_3 = f(x_i + h, y_i - hS_1 + 2hS_2)$$

First-order Ordinary Differential Equations

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- Example 11 – Third-order RK Method

Solve the following differential equation using the third-order Runge-Kutta method for $1 \leq x \geq 2$ with step size of $h = 0.1$:

$$\frac{dy}{dx} - y\left(x^2 + 1\right) = 0 \qquad \text{such that } y = 2 \text{ at } x = 1$$

and compare your result with the exact solution as given by:

$$y = e^{\frac{x^3}{3} + x - 0.640186}$$

First-order Ordinary Differential Equations

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- Example 11 (cont'd) – Third-order RK Method

<u>$i = 0$:</u>

$$\frac{dy}{dx} = y\left(x^2 + 1\right) = f\left(x_i, y_i\right), \quad x_0 = 1, y_0 = 2$$

$$S_1 = f(1,2) = (2)\left(1^2 + 1\right) = 4$$

$$x_0 + 0.5h = 1 + 0.5(0.1) = 1.05$$

$$y_0 + 0.5hS_1 = 2 + 0.5(0.1)(4) = 2.2$$

First-order Ordinary Differential Equations

- Example 11 (cont'd) – Third-order RK Method

$$S_2 = f(1.05, 2.2) = \left.\frac{dy}{dx}\right|_{\substack{x=1.05\\y=2.2}} = 2.2\left[(1.05)^2 + 1\right] = 4.6255$$

$$x_0 + h = 1 + 0.1 = 1.1$$

$$y_0 - hS_1 + 2hS_2 = 2 - 0.1(4) + 2(0.1)(4.6255) = 2.5251$$

First-order Ordinary Differential Equations

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- Example 11 (cont'd) – Third-order RK Method

$$S_3 = f(1.1, 2.5251) = \left.\frac{dy}{dx}\right|_{\substack{x=1.1\\y=2.5251}}$$

$$= (2.5251)\left[(1.1)^2 + 1\right] = 5.580471$$

Therefore

$$y_1 = 2 + \frac{0.1}{6}\left[4 + 4(4.6255) + 5.580471\right] = \underline{2.468041}$$

First-order Ordinary Differential Equations

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- Example 11 (cont'd) – Third-order RK Method

$i = 1$:

$$y_2 = y_1 + \frac{h}{6}[S_1 + 4S_2 + S_3]$$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y_1 = 2.468041$$

$$S_1 = 5.454371, \quad S_2 = 6.365414, \quad S_3 = 7.79476$$

Therefore,

$$y_2 = 2.468041 + \frac{0.1}{6}[S_1 + 4S_2 + S_3] = 3.113266$$

First-order Ordinary Differential Equations

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- Example 11 (cont'd) – Third-order RK Method

$\frac{dy}{dx} - y(x^2 + 1) = 0$ The exact solution is given by: $y = e^{\frac{x^3}{3} + x - 0.640186}$

such that $y(1) = 2$ $1.0 \le x \le 2.0$

h = 0.1

i	x	S_1	S_2	S_3	y (R-K third-order)	y (exact)	% error
0	1	4.000000	-	-	-	2	-
1	1.1	5.454371	4.625500	5.580471	2.468041	2.468179	0.005587
2	1.2	7.596370	6.365414	7.797476	3.113266	3.113707	0.014157
3	1.3	10.820228	8.951030	11.146917	4.022390	4.023481	0.027112
4	1.4	15.787307	12.880200	16.328564	5.333550	5.336023	0.046359
5	1.5	23.634890	18.996343	24.550785	7.272274	7.277694	0.074469
6	1.6	36.370522	28.764798	37.955810	10.216439	10.228187	0.114860
7	1.7	57.637210	44.800157	60.448336	14.816764	14.842292	0.171997
8	1.8	94.241231	71.900660	99.356660	22.226705	22.282776	0.251631
9	1.9	159.295038	119.136697	168.863939	34.554238	34.679451	0.361059
10	2	278.888353	204.197449	297.321121	55.777671	56.063258	0.509403

First-order Ordinary Differential Equations

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- Fourth-order Runge-Kutta Method

$$y_{i+1} = y_i + \frac{h}{6}\left[S_1 + 2S_2 + 2S_3 + S_4\right] \qquad (23)$$

$$S_1 = f(x_i, y_i)$$
$$S_2 = f(x_i + 0.5h, y_i + 0.5hS_1)$$
$$S_3 = f(x_i + 0.5h, y_i + 0.5hS_2)$$
$$S_4 = f(x_i + h, y_i + hS_3)$$

First-order Ordinary Differential Equations

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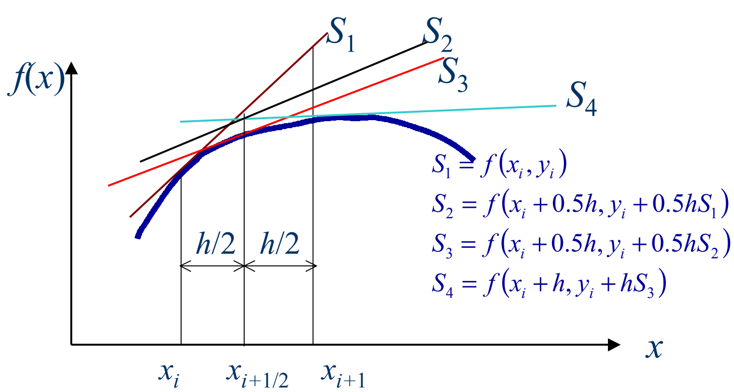
- Fourth-order Runge-Kutta Method



First-order Ordinary Differential Equations

- Example 12 – Fourth-order RK Method

Solve the following differential equation of Example 11 using the third-order Runge-Kutta method for $1 \leq x \geq 2$ with step size of $h = 0.1$:

$$\frac{dy}{dx} - y\left(x^2 + 1\right) = 0 \qquad \text{such that } y = 2 \text{ at } x = 1$$

and compare your result with the exact solution as given by:

$$y = e^{\frac{x^3}{3} + x - 0.640186}$$

First-order Ordinary Differential Equations

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- Example 12 (cont'd) – Fourth-order RK Method

$\underline{i = 0:}$

$$\frac{dy}{dx} = y\left(x^2 + 1\right) = f\left(x_i, y_i\right), \qquad x_0 = 1, y_0 = 2$$

$$y_{i+1} = y_i + \frac{h}{6}\left[S_1 + 2S_2 + 2S_3 + S_4\right]$$

$$y_1 = y_0 + \frac{h}{6}\left[S_1 + 2S_2 + 2S_3 + S_4\right]$$

First-order Ordinary Differential Equations

- Example 12 (cont'd) – Fourth-order RK Method

$$S_1 = f(1,2) = 2(1^2 + 1) = 4$$
$$x_0 + 0.5h = 1 + 0.5(0.1) = 1.05$$
$$y_0 + 0.5hS_1 = 2 + 0.5(0.1)(4) = 2.2$$
$$S_2 = f(1.05, 2.2) = 2.2\left[(1.05)^2 + 1\right] = 4.6255$$
$$x_0 + 0.5h = 1 + 0.5(0.1) = 1.05$$
$$y_0 + 0.5hS_2 = 2 + 0.5(0.1)(4.6255) = 2.231275$$

First-order Ordinary Differential Equations

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- Example 12 (cont'd) – Fourth-order RK Method

$$S_3 = f(1.05, 2.231275) = 2.231275\left[(1.05)^2 + 1\right] = 4.691256$$
$$x_0 + h = 1 + 0.1 = 1.1$$
$$y_0 + hS_3 = 2 + 0.1(4.691256) = 2.469126$$
$$S_4 = f(1.1, 2.469126) = 2.469126\left[(1.1)^2 + 1\right] = 5.456768$$

Therefore,

$$y_1 = 2 + \frac{0.1}{6}\left[4 + 2(4.6255) + 2(4.691256) + 5.456768\right]$$
$$= \underline{2.468171}$$

First-order Ordinary Differential Equations

- Example 12 (cont'd) – Fourth-order RK Method

<u>$i = 1$:</u>

$$\frac{dy}{dx} = y\left(x^2+1\right) = f\left(x_i, y_i\right), \quad x_0 = 1.1, y_0 = 2.468171$$

$$y_{i+1} = y_i + \frac{h}{6}\left[S_1 + 2S_2 + 2S_3 + S_4\right]$$

$$y_2 = y_1 + \frac{h}{6}\left[S_1 + 2S_2 + 2S_3 + S_4\right]$$

First-order Ordinary Differential Equations

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- Example 12 (cont'd) – Fourth-order RK Method

$$S_1 = f(1.1, 2.468171) = 2.468171\left[(1.1)^2 + 1\right] = 5.454659$$

$$x_1 + 0.5h = 1.1 + 0.5(0.1) = 1.15$$

$$y_1 + 0.5hS_1 = 2.468171 + 0.5(0.1)(5.454659) = 2.740904$$

$$S_2 = f(1.15, 2.740904) = 2.740904\left[(1.15)^2 + 1\right] = 6.365749$$

$$x_1 + 0.5h = 1.1 + 0.5(0.1) = 1.15$$

$$y_2 + 0.5hS_2 = 2.468171 + 0.5(0.1)(6.365749) = 2.786458$$

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First-order Ordinary Differential Equations

- Example 12 (cont'd) – Fourth-order RK Method

$$S_3 = f(1.15, 2.786458) = 2.2786458\left[(1.15)^2 + 1\right] = 6.47155$$

$$x_1 + h = 1.1 + 0.1 = 1.2$$

$$y_1 + hS_3 = 2.468171 + 0.1(6.47155) = 3.115326$$

$$S_4 = f(1.2, 3.115326) = 3.115326\left[(1.2)^2 + 1\right] = 7.601395$$

Therefore,

$$y_2 = 2.468171 + \frac{0.1}{6}\left[5.454659 + 2(6.365749) + 2(6.47155) + 7.601395\right]$$

$$= \underline{3.113682}$$

First-order Ordinary Differential Equations

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- Example 12 (cont'd) – Fourth-order RK Method

$$\frac{dy}{dx} - y(x^2 + 1) = 0$$

The exact solution is given by: $y = e^{\frac{x^3}{3} + x - 0.640186}$

such that $y(1) = 2$ $\quad 1.0 \le x \le 2.0$

$h =$ 0.1

i	x	S_1	S_2	S_3	S_4	y (R-K fourth-order)	y (exact)	% error
0	1	4.000000	-	-	-	-	2	-
1	1.1	5.454659	4.625500	4.691256	5.456768	2.468171	2.468179	0.000315
2	1.2	7.597404	6.365750	6.471551	7.601396	3.113690	3.113707	0.000547
3	1.3	10.823066	8.952248	9.125837	10.830676	4.023445	4.023481	0.000893
4	1.4	15.794402	12.883578	13.174367	15.809009	5.335947	5.336023	0.001436
5	1.5	23.651964	19.004881	19.502907	23.680271	7.277527	7.277694	0.002284
6	1.6	36.411038	28.785577	29.658933	36.466578	10.227820	10.228187	0.003591
7	1.7	57.733295	44.850063	46.420777	57.843901	14.841464	14.842292	0.005579
8	1.8	94.470880	72.020523	74.922616	94.694996	22.280868	22.282776	0.008562
9	1.9	159.851518	119.427012	124.945437	160.314648	34.674950	34.679451	0.012981
10	2	280.261797	204.910792	215.730650	281.240073	56.052359	56.063258	0.019441

First-order Ordinary Differential Equations

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- Least-Squares Method
  - In the previous methods, the solution is obtained as a series of points using the step size or interval h.
  - The least-squares method provides a function that can be used to approximate the value of the dependent variable y for any given value of the independent variable x.

First-order Ordinary Differential Equations

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- Least-Squares Method

  Procedure for Deriving Least-Squares Function

  1. For a given ordinary differential equation, assume that the solution is a differentiable function such as given by the following polynomial:

$$\hat{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \qquad (24)$$

First-order Ordinary Differential Equations

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- Least-Squares Method
  2. Use the boundary condition of the ordinary differential equation to evaluate one of the coefficients of the assumed function or to provide a condition toward the solution of the coefficients.
  3. Define the least-squares objective function *F*, which is the integral of the errors, *e*, squared.

$$F = \int_x e^2 dx \qquad (25)$$

First-order Ordinary Differential Equations

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- Least-Squares Method

  The error is defined as the difference between the derivatives for the assumed function (EQ. 24) and the actual function, that is

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} \qquad (26)$$

  4. Find the minimum of F with respect to the unknowns $b_1, b_2, \ldots, b_n$ of the assumed function of Eq. 24 by taking the derivatives of Eq. 25 with respect to the unknowns and setting the derivatives equal to zero, that is

First-order Ordinary Differential Equations

- Least-Squares Method

$$\frac{\partial F}{\partial b_i} = \int_{\text{all } x} 2e \frac{\partial e}{\partial b_i} dx = 0 \qquad (27)$$

5. The integrals of Equation 27 are called the *normal equations*. The solution of the normal equations yields values of the unknowns coefficients $b_1, b_2, \ldots, b_n$.

First-order Ordinary Differential Equations

- Example 13 – Least-Squares Method

Solve the following differential equation by finding an expression for the dependent variable y using the least-squares method and compare with the true function $y = e^{x^2/2}$:

$$\frac{dy}{dx} - xy = 0 \qquad \text{such that } y = 1 \text{ at } x = 0$$

Use a quadratic model for your expression for $0 \leq x \geq 1$.

First-order Ordinary Differential Equations

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- Example 13 (cont'd)– Least-Squares Method

$$\hat{y} = b_0 + b_1 x + b_2 x^2$$

Using the boundary condition:

$$1 = b_0 + b_1(0) + b_2(0)^2$$

Therefore,

$$b_0 = 1$$

First-order Ordinary Differential Equations

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- Example 13 (cont'd)– Least-Squares Method

The derivative is

$$\frac{d\hat{y}}{dx} = b_1 + 2b_2 x$$

The error function is

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} = b_1 + 2b_2 x - xy$$

$$= b_1 + 2b_2 x - x\left(1 + b_1 x + b_2 x^2\right)$$

First-order Ordinary Differential Equations

- Example 13 (cont'd)– Least-Squares Method

$$e = b_1\left(1 - x^2\right) + b_2\left(2x - x^3\right) - x$$

The derivatives can be computed for b_1 and b_2 as follows:

$$\frac{\partial e}{\partial b_1} = 1 - x^2$$

$$\frac{\partial e}{\partial b_2} = 2x - x^3$$

First-order Ordinary Differential Equations

- Example 13 (cont'd)– Least-Squares Method

Using Eq. 27, and

$$\frac{\partial e}{\partial b_1} = 1 - x^2$$

$$e = b_1\left(1 - x^2\right) + b_2\left(2x - x^3\right) - x$$

$$\frac{\partial F}{\partial b_i} = \int_{\text{all } x} 2e \frac{\partial e}{\partial b_i} dx = 0$$

$$2\int_0^1 \left[b_1\left(1 - x^2\right) + b_2\left(2x - x^3\right) - x\right]\left(1 - x^2\right)dx = 0$$

$$\frac{8b_1}{15} + \frac{5b_2}{12} = \frac{1}{4}$$ ← Normal equation 1

First-order Ordinary Differential Equations

- Example 13 (cont'd)– Least-Squares Method

$$\frac{\partial e}{\partial b_2} = 2x - x^3$$

Using Eq. 27, and

$$e = b_1(1-x^2) + b_2(2x-x^3) - x$$

$$\frac{\partial F}{\partial b_i} = \int_{\text{all } x} 2e \frac{\partial e}{\partial b_i} dx = 0$$

$$2\int_0^1 \left[b_1(1-x^2) + b_2(2x-x^3) - x\right](2x-x^3)dx = 0$$

$$\frac{9b_1}{20} + \frac{71b_2}{105} = \frac{7}{15}$$ ← Normal equation 2

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- Example 13 (cont'd)– Least-Squares Method

Solving the normal equations simultaneously, yields the following approximating function for the y

$$\hat{y} = 1 - 0.14669x + 0.78776x^2$$

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- Example 13 (cont'd)– Least-Squares Method

x	Numerical *y* value	True *y* value	% error
0	1	1	0
0.2	1.002172	1.020201	1.77
0.4	1.067366	1.083287	1.47
0.6	1.19558	1.197217	0.14
0.8	1.386814	1.377128	0.70
1	1.64107	1.648721	0.46

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- Galerkin Method
  - This method is similar to the least-squares method in a sense that an approximating function is necessary.
  - The method uses the error function of Eq. 26 and approximating equation such as Eq. 24

$$e = \frac{d\hat{y}}{dx} - \frac{dy}{dx} \qquad \hat{y} = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

First-order Ordinary Differential Equations

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- Galerkin Method

  The normal equations are computed are computed from the following expression:

$$\int_x w_i e\,dx = 0 \qquad \text{for } i = 1,2,\cdots,n \qquad (28)$$

n = number of unknowns

$w_i = \frac{d\hat{y}}{db_i}$ = weighting factor

First-order Ordinary Differential Equations

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- Example 14 – Galerkin Method

  Solve the following differential equation using Galerkin method with a quadratic model and for $1 \le x \ge 1.5$:

$$\frac{dy}{dx} - 5x + 2 = 0 \quad \text{such that } y = 1 \text{ at } x = 1$$

Assess the accuracy of the model as compared to the true solution.

First-order Ordinary Differential Equations

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- Example 14 (cont'd)– Galerkin Method

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5x - 2$$

Quadratic Model: $\hat{y} = b_0 + b_1 x + b_2 x^2$

Using the boundary condition

$$\hat{y} = 1 = b_0 + b_1(1) + b_2(1)^2$$

yields $b_0 = 1 - b_1 - b_2$. Thus, the quadratic model is

$$\hat{y} = 1 - b_1 + b_1 x + b_2 x^2 - b_2$$

and

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = b_1 + 2b_2 x$$

First-order Ordinary Differential Equations

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- Example 14 (cont'd)– Galerkin Method

$$e = \frac{\mathrm{d}\hat{y}}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = b_1 + 2b_2 x - (5x - 2) = b_1 + 2b_2 x - 5x + 2$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_1} = x - 1$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_2} = x^2 - 1$$

Using Eq. 8-57 of the textbook, the following results can be obtained:

$$\int_1^x \left[b_1 + 2b_2 x - 5x + 2\right](x - 1)\,\mathrm{d}x = 0$$

$$\int_1^x \left[b_1 + 2b_2 x - 5x + 2\right](x^2 - 1)\,\mathrm{d}x = 0$$

First-order Ordinary Differential Equations

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- Example 14 (cont'd)– Galerkin Method

Using $x = 1.5$ as the upper limit, above two equations yield the following two normal equations:

$$0.125000b_1 + 0.333333b_2 - 0.583333 = 0$$
$$0.291667b_1 + 0.781250b_2 - 1.369792 = 0$$

The solution of the normal equations is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2.5 \end{bmatrix}$$

Thus, the quadratic approximation is given by

$$\hat{y} = 1 - (-2) - 2x + 2.5x^2 - 2.5$$
$$= 0.5 - 2x + 2.5x^2 \quad \Rightarrow \text{(identical to true solution)}$$

First-order Ordinary Differential Equations

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- Example 14 (cont'd)– Galerkin Method

For $x = 1.1$, $\hat{y} = 1.325$

The following table summarizes the estimated values of the function $\hat{y}$ for the range of x = 1 to 1.5 in an increment of 0.1. In this problem, the Galerkin method gave the true solution, hence no error is generated.

x	Estimated $y(x)$	True $y(x)$	error(%)
1	1	1	0.00000
1.1	1.325	1.325	0.00000
1.3	2.125	2.125	0.00000
1.5	3.125	3.125	0.00000