CHAPTER 8b. DIFFERENTIAL EQUATIONS

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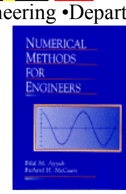

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ENCE 203 - Computation Methods in Civil Engineering II

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Taylor Series Expansion

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■ Fundamental Case

Assume that the problem is a first-order differential equation of the form

$$\frac{dy}{dx} = f(x) \qquad \text{subject to } y = y_0 \text{ at } x = x_0 \quad (5)$$

If the variables are separated and the integration is carried out on both sides, then

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x)dx \qquad (6)$$

Taylor Series Expansion

■ Fundamental Case

Or

$$y\Big|_{y_0}^{y} = \int_{x_0}^{x} f(x)\,dx$$

$$y - y_0 = \int_{x_0}^{x} f(x)\,dx$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x)\,dx \qquad (7)$$

Taylor Series Expansion

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■ Fundamental Case

– Recall Taylor's Series Expansion

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!} f^{(2)}(x_0) + \frac{h^3}{3!} f^{(3)}(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + R_{n+1} \qquad (8)$$

where

x_0 = base value or starting value

x = the point at which the value of the function is needed

$h = x – x_0$ = distance between x_0 and x (step size)

$n!$ = factorial of $n = n(n-1)(n – 2)\ldots1$

$f^{(n)}$ = indicates the n^{th} derivative of the function $f(x)$

R_{n+1} = the remainder of Taylor series expansion

Taylor Series Expansion

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■ Fundamental Case

– Taylor's Series Expansion

Eq. (8) can be expressed as

$$g(x) = g(x_0) + (x - x_0)g'(x_0) + \frac{(x - x_0)^2}{2!}g''(x_0) + \frac{(x - x_0)^3}{3!}g'''(x_0) + \cdots \qquad (9a)$$

Or

$$y(x) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{x=x_0} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=x_0} + \frac{(x - x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=x_0} + \cdots \qquad (9b)$$

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■ Fundamental Case

– Taylor's Series Expansion

- Comparing Eq. 9b and Eq. 7, we can evaluate the integral of Eq. 7 by a Taylor Series Expansion:

$$y(x) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{x=x_0} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=x_0} + \frac{(x - x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=x_0} + \cdots$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x)$$

Taylor Series Expansion

■ Fundamental Case

– Taylor's Series Expansion

- In view of the integral of the second equation, the comparison implies that

$$\int_{x_0}^{x} f(x) = (x - x_0)\left.\frac{dy}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=x_0} + \cdots$$

- Therefore, Equations 9b and 7 can be used to solve first order equations.

Taylor Series Expansion

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■ Example 1 - Taylor Series Expansion

Solve the following differential equation using Taylor's series expansion:

$$\frac{dy}{dx} = 3x^2 \qquad \text{such that } y = 1 \text{ at } x = 1$$

$x_0 = 1$

$y_0 = 1$

The higher-order derivative can be obtained as follows:

$$\frac{d^2x}{dx^2} = 6x \qquad \frac{d^3y}{dx^3} = 6 \qquad \frac{d^ny}{dx^n} = 0 \quad \text{for } n \ge 4$$

Taylor Series Expansion

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■ Example 1 (cont'd) - Taylor Series Expansion

Using the Taylor's series expansion of Eq. 9b gives

$$y(x) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{x=x_0}$$

$$y(x) = 1 + (x-1)(3x_0^2) + \frac{(x-1)^2}{2}(6x_0) + \frac{(x-1)^3}{6}(6)$$

Taylor Series Expansion

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■ Example 1(cont'd) - Taylor Series Expansion

Substituting for x_0 = 1 in the last equation, gives the solution of the differential equation

$$y(x) = 1 + (x-1)(3) + 3(x-1)^2 + (x-1)^3$$

Taylor Series Expansion

■ Example 1(cont'd) $y(x) = 1 + (x-1)(3) + 3(x-1)^2 + (x-1)^3$

	$y(x)$			
x	One Term	Two Terms	Three Terms	Four Terms
1	1	1	1	1
1.1	1	1.3	1.33	1.331
1.2	1	1.6	1.72	1.728
1.3	1	1.9	2.17	2.197
1.4	1	2.2	2.68	2.744
1.5	1	2.5	3.25	3.375
1.6	1	2.8	3.88	4.096
1.7	1	3.1	4.57	4.913
1.8	1	3.4	5.32	5.832
1.9	1	3.7	6.13	6.859
2	1	4	7	8

Taylor Series Expansion

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■ Example 1(cont'd)

The exact solution can be obtained as follows:

$$\int_1^y dy = \int_1^x 3x^2 dx$$

$$y - 1 = \left.\frac{3x^3}{3}\right|_1^x = \left.x^3\right|_1^x$$

$$y - 1 = x^3 - 1$$

$$y = x^3$$

Taylor Series Expansion

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■ Example 1(cont'd) $y = x^3$

	$y(x)$				
x	One Term	Two Terms	Three Terms	Four Terms	TRUE
1	1	1	1	1	1
1.1	1	1.3	1.33	1.331	1.331
1.2	1	1.6	1.72	1.728	1.728
1.3	1	1.9	2.17	2.197	2.197
1.4	1	2.2	2.68	2.744	2.744
1.5	1	2.5	3.25	3.375	3.375
1.6	1	2.8	3.88	4.096	4.096
1.7	1	3.1	4.57	4.913	4.913
1.8	1	3.4	5.32	5.832	5.832
1.9	1	3.7	6.13	6.859	6.859
2	1	4	7	8	8

Taylor Series Expansion

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■ Example 1(cont'd) – Taylor Series

– Examining the table, we notice that the Taylor's series solution for this example gives no error when using 4 terms.

– This is because, the derivatives beyond the third equal to zero.

– In this case, Taylor's series expansion provides the true solution when all the terms are used.

Taylor Series Expansion

■ General Case

Assume that the problem is a first-order ordinary differential equation of the following form:

$$\frac{dy}{dx} = f(x, y) \qquad \text{subject to } y = y_0 \text{ at } x = x_0$$

In this case the Taylor series expansion is

$$g(x) = g(x_0, y_0) + (x - x_0)g'(x_0, y_0) + \frac{(x - x_0)^2}{2!}g''(x_0, y_0) + \frac{(x - x_0)^3}{3!}g'''(x_0, y_0) + \cdots \qquad (10)$$

Taylor Series Expansion

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■ General Case

Or in equivalent form, Taylor series can be given as

$$y(x, y) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} + \frac{(x - x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0\\y=y_0}} + \frac{(x - x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{\substack{x=x_0\\y=y_0}} + \cdots \qquad (11)$$

■ Example 2 - Taylor Series Expansion

Solve the following differential equation using Taylor's series expansion:

$$\frac{dy}{dx} = 3x^2 y \qquad \text{such that } y = 1 \text{ at } x = 1$$

$x_0 = 1$

$y_0 = 1$

The higher-order derivatives can be obtained as follows:

$$\frac{d^2y}{dx^2} = 6xy + 3x^2\frac{dy}{dx}$$

Taylor Series Expansion

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■ Example 2 - Taylor Series Expansion

$$\frac{d^2y}{dx^2} = 6xy + 3x^2\frac{dy}{dx} = 6xy + 3x^2\left(3x^2y\right)$$
$$= 6xy + 9x^4 y \quad \Longrightarrow \quad \left.\frac{d^2y}{dx^2}\right|_{\substack{x_0=1\\y_0=1}} = 15$$

$$\frac{d^3y}{dx^3} = 6y + 6x\frac{dy}{dx} + 36x^3y + 9x^4\frac{dy}{dx}$$
$$= 6y + 6x\left(3x^2y\right) + 36x^3y + 9x^4\left(3x^2y\right) \quad \Longrightarrow \quad \left.\frac{d^3y}{dx^3}\right|_{\substack{x_0=1\\y_0=1}} = 87$$
$$= 6y + 6x\left(3x^2y\right) + 36x^3y + 27x^6y$$

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Taylor Series Expansion

■ Example 2 (cont'd) - Taylor Series Expansion

Using the Taylor's series expansion of Eq. 11 gives

$$y(x,y) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{\substack{x=x_0\\y=y_0}} + \frac{(x-x_0)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{\substack{x=x_0\\y=y_0}}$$

$$y(x,y) = 1 + (x-1)(3) + \frac{(x-1)^2}{2}(15) + \frac{(x-1)^3}{6}(87)$$

Taylor Series Expansion

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■ Example 2 (cont'd) - Taylor Series Expansion

Substituting for x_0 = 1 and $y_0 = 1$ in the last equation, gives the solution of the differential equation for four terms as

$$y(x) = 1 + (x-1)(3) + 7.5(x-1)^2 + 14.5(x-1)^3$$

Taylor Series Expansion

- Example 2 (cont'd) - Taylor Series Expansion $y(x)=1+(x-1)(3)+7.5(x-1)^2+14.5(x-1)^3$

	$y(x)$			
x	One Term	Two Terms	Three Terms	Four Terms
1	1	1	1	1
1.1	1	1.3	1.375	1.3895
1.2	1	1.6	1.9	2.016
1.3	1	1.9	2.575	2.9665
1.4	1	2.2	3.4	4.328
1.5	1	2.5	4.375	6.1875
1.6	1	2.8	5.5	8.632
1.7	1	3.1	6.775	11.7485
1.8	1	3.4	8.2	15.624
1.9	1	3.7	9.775	20.3455
2	1	4	11.5	26

Taylor Series Expansion

- Example 2 (cont'd)

The exact solution can be obtained as follows:

$$\frac{dy}{dx}=3x^2y \Rightarrow \int_1^y \frac{dy}{y}=\int_1^x 3x^2dx$$

$$\ln y-\ln 1=\left.\frac{3x^3}{3}\right|_1^x=\left.x^3\right|_1^x$$

($\ln 1$ = 0)

$$\ln y=x^3-1$$

$$y=e^{x^3-1}$$

Taylor Series Expansion

- Example 2 (cont'd) - Taylor Series Expansion $y = e^{x^3 - 1}$

	$y(x)$				
x	One Term	Two Terms	Three Terms	Four Terms	TRUE
1	1	1	1	1	1
1.1	1	1.3	1.375	1.390	1.392
1.2	1	1.6	1.9	2.016	2.071
1.3	1	1.9	2.575	2.967	3.310
1.4	1	2.2	3.4	4.328	5.720
1.5	1	2.5	4.375	6.188	10.751
1.6	1	2.8	5.5	8.632	22.109
1.7	1	3.1	6.775	11.749	50.049
1.8	1	3.4	8.2	15.624	125.462
1.9	1	3.7	9.775	20.346	350.374
2	1	4	11.5	26.000	1096.633

Taylor Series Expansion

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- Example 2 (cont'd) - Taylor Series Expansion
  - Viewing the results of the solution based on Taylor series expansion, we notice that as the number of terms increases, the accuracy of the solution improves.
  - Also, as the step size decreases, the accuracy of the solution improves.

First-order Ordinary Differential Equations

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- Euler's Method
  - As we noticed in the previous example, in some cases the derivatives are not easily computed.
  - Therefore, the Taylor series of Eqs. 9, 10, and 11 can be truncated so that only the term with the first derivative is used.

First-order Ordinary Differential Equations

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- Euler's Method
  - The value of the dependent variable $y = g(x)$ can be computed using

$$g(x) = g(x_0, y_0) + (x - x_0)g'(x_0, y_0) + e \qquad (12a)$$

or

$$y(x) = y_0 + (x - x_0)\left.\frac{dy}{dx}\right|_{\substack{x=x_0\\y=y_0}} + e \qquad (12b)$$

First-order Ordinary Differential Equations

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- Euler's Method
  - For better accuracy, $(x - x_0)$ should be made small.
  - Notice that $(x - x_0) = \Delta x = h$
  - The above equations can be rewritten in a more compact form for computer implementation as

$$y_{i+1} = y_i + hf(x_i, y_i)$$

First-order Ordinary Differential Equations

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- Euler's Method

  The iterative procedure for basic Euler's method is given by

$$y_{i+1} = y_i + hf(x_i, y_i) \qquad (13)$$

where

$$h = x - x_0$$

$$f(x_0, y_0) = g'(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x = x_0 \\ y = y_0}}$$

First-order Ordinary Differential Equations

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■ Example 3 – Euler's Method

Solve the following differential equation for $0 \le x \ge 1$ using a step size of $h = 0.1$:

$$\frac{dy}{dx} - \frac{1}{2}y = 0 \qquad \text{such that } y = 1 \text{ at } x = 0$$

Here we have

$$y(0) = 1 \qquad \text{or} \qquad x_0 = 0, \quad y_0 = 1$$

First-order Ordinary Differential Equations

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■ Example 3 (cont'd) – Euler's Method

First Iteration ($i = 0$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$x_0 = 0, \quad y_0 = 1, \text{ and } \quad h = 0.1$$

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x_0=0\\ y_0=1}} = \frac{1}{2}y = \frac{1}{2}(1) = \frac{1}{2}$$

$$\therefore y_1 = 1 + 0.1\left(\frac{1}{2}\right) = 1 + 0.05 = 1.05$$

First-order Ordinary Differential Equations

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■ Example 3 (cont'd) – Euler's Method

Second Iteration ($i = 1$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$x_1 = 0.1, \quad y_1 = 1.05, \text{ and } \quad h = 0.1$$

$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=0.1 \\ y_0=1.05}} = \frac{1}{2}y = \frac{1}{2}(1.05) = 0.5250$$

$$\therefore y_2 = 1.05 + 0.1(0.5250) = 1.05 + 0.0525 = 1.1025$$

First-order Ordinary Differential Equations

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■ Example 3 (cont'd) – Euler's Method

Third Iteration ($i = 2$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$x_2 = 0.2, \quad y_2 = 1.1025, \text{ and } \quad h = 0.1$$

$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1=0.2 \\ y_0=1.1025}} = \frac{1}{2}y = \frac{1}{2}(1.1025) = 0.55125$$

$$\therefore y_3 = 1.1025 + 0.1(0.55125) = 1.157625$$

First-order Ordinary Differential Equations

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■ Example 3 (cont'd) – Euler's Method

– See the spreadsheet output in the next viewgraph for the rest of the iterations.

– Expression for the exact solution can be obtained as follows:

$$x_0 = 0 \qquad \ln 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2}y \Rightarrow \int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{1}{2}dx$$

$$\ln y - \ln 1 = \frac{1}{2}(x - x_0) \quad \Rightarrow \quad \ln y = \frac{1}{2}x$$

$$y = e^{\frac{x}{2}}$$

First-order Ordinary Differential Equations

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■ Example 3 (cont'd) – Euler's Method

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
0	0	0	1	0.5		1	
1	0.1	0.1	1.050000	0.525000	1.050000	1.051271	0.12
2	0.2	0.2	1.102500	0.551250	1.102500	1.105171	0.24
3	0.3	0.3	1.157625	0.578813	1.157625	1.161834	0.36
4	0.4	0.4	1.215506	0.607753	1.215506	1.221403	0.48
5	0.5	0.5	1.276282	0.638141	1.276282	1.284025	0.60
6	0.6	0.6	1.340096	0.670048	1.340096	1.349859	0.72
7	0.7	0.7	1.407100	0.703550	1.407100	1.419068	0.84
8	0.8	0.8	1.477455	0.738728	1.477455	1.491825	0.96
9	0.9	0.9	1.551328	0.775664	1.551328	1.568312	1.08
10	1	1	1.628895	0.814447	1.628895	1.648721	1.20

$$\text{True Function}: y = e^{\frac{1}{2}x}$$

First-order Ordinary Differential Equations

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■ Example 4 - Taylor Series Expansion

Solve the following differential equation using Euler's method for $1 \leq x \geq 2$ with a step size of *h* = 0.1:

$$\frac{dy}{dx} = 3x^2 \qquad \text{such that } y = 1 \text{ at } x = 1$$

$$y(1) = 1 \qquad \text{or} \qquad x_0 = 1$$
$$y_0 = 1$$

First-order Ordinary Differential Equations

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■ Example 4 (cont'd) – Euler's Method

First Iteration ($i = 0$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_1 = y_0 + hf(x_0, y_0)$$
$$x_0 = 1, \quad y_0 = 1, \text{ and } \quad h = 0.1$$
$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{\substack{x_0=1 \\ y_0=1}} = 3x^2 = 3(1)^2 = 3$$
$$\therefore y_1 = 1 + 0.1(3) = 1 + 0.3 = 1.30$$

First-order Ordinary Differential Equations

- Example 4 (cont'd) – Euler's Method

  Second Iteration ($i = 1$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$x_1 = 1.1, \quad y_1 = 1.30, \text{ and } \quad h = 0.1$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1 = 0.1 \\ y_0 = 1.30}} = 3x^2 = 3(1.1)^2 = 3.630$$
$$\therefore y_2 = 1.30 + 0.1(3.630) = 1.6630$$

First-order Ordinary Differential Equations

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- Example 4 (cont'd) – Euler's Method

  Third Iteration ($i = 2$):

$$y_{i+1} = y_i + hf(x_i, y_i)$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$x_2 = 1.2, \quad y_2 = 1.663, \text{ and } \quad h = 0.1$$
$$f(x_1, y_1) = \left.\frac{dy}{dx}\right|_{\substack{x_1 = 1.2 \\ y_0 = 1.663}} = 3x^2 = 3(1.2)^2 = 4.320$$
$$\therefore y_3 = 1.663 + 0.1(0.4.320) = 2.095$$

First-order Ordinary Differential Equations

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- Example 4 (cont'd) – Euler's Method
  - See the spreadsheet output in the next viewgraph for the rest of the iterations.
  - Expression for the exact solution can be obtained as follows:

$$x_0 = 1 \quad y_0 = 1$$

$$\int_1^y dy = \int_1^x 3x^2 dx$$

$$y - 1 = \left.\frac{3x^3}{3}\right|_1^x = \left.x^3\right|_1^x$$

$$y - 1 = x^3 - 1$$

$$y = x^3$$

First-order Ordinary Differential Equations

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- Example 4 (cont'd) – Euler's Method

i	x	x_i	y_i	$f(x_i, y_i)$	y (Euler)	y (True)	% Error
0	1	1	1	3		1	
1	1.1	1.1	1.300000	3.63	1.300000	1.331	2.33
2	1.2	1.2	1.663000	4.32	1.663000	1.728	3.76
3	1.3	1.3	2.095000	5.07	2.095000	2.197	4.64
4	1.4	1.4	2.602000	5.88	2.602000	2.744	5.17
5	1.5	1.5	3.190000	6.75	3.190000	3.375	5.48
6	1.6	1.6	3.865000	7.68	3.865000	4.096	5.64
7	1.7	1.7	4.633000	8.67	4.633000	4.913	5.70
8	1.8	1.8	5.500000	9.72	5.500000	5.832	5.69
9	1.9	1.9	6.472000	10.83	6.472000	6.859	5.64
10	2	2	7.555000	12	7.555000	8	5.56

True Function : $y = x^3$