CHAPTER 7d. DIFFERENTIATION AND INTEGRATION

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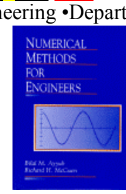

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Numerical Integration

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- Example 3

  Evaluate the following integral using the trapezoidal rule. Use interval widths of 1, 0.5, and 0.25, and compare your results with the true value of *I* = -0.346078:

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx$$

Numerical Integration

- Example 3 (cont'd)

For interval width of 1, $n = \frac{3-1}{1} + 1 = 2 + 1 = 3$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	2	3
$f(x)$	0.145310	-0.198424	-0.422561

Numerical Integration

- Example 3 (cont'd)

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1}^{3-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$= (2\text{-}1)\frac{(-0.198424 + 0.145310)}{2} + (3-2)\frac{(-0.422561 + (-0.198424))}{2}$$

$$= \text{-}0.026557 + (\text{-}0.3104925)$$

$$= \underline{\text{-}0.3370495}$$

x	1	2	3
$f(x)$	0.145310	-0.198424	-0.422561

Numerical Integration

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■ Example 3 (cont'd)

For interval width of 0.5, $n = \frac{3-1}{0.5} + 1 = 4 + 1 = 5$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	1.5	2	2.5	3
$f(x)$	0.145310	0.029035	-0.198424	-0.379833	-0.422561

Numerical Integration

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■ Example 3 (cont'd)

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1}^{5-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$= (1.5\text{-}1)\frac{(0.029035 + 0.145310)}{2} + (2 - 1.5)\frac{(-0.198424 + 0.029035)}{2}$$

$$+ (2.5\text{-}2)\frac{(-0.379833 - 0.198424)}{2} + (3\text{-}2.5)\frac{(-0.422561 - 0.379833)}{2}$$

$$= 0.04358607 - 0.0423473 - 0.1445642 - 0.2005986$$

$$= \underline{-0.343924}$$

x	1	1.5	2	2.5	3
$f(x)$	0.145310	0.029035	-0.198424	-0.379833	-0.422561

Numerical Integration

■ Example 3 (cont'd)

For interval width of 0.25, $n = \dfrac{3-1}{0.25} + 1 = 8 + 1 = 9$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$f(x)$	0.145310	0.109722	0.029035	-0.080816	-0.198424	-0.303224	-0.379833	-0.420008	-0.422561

Numerical Integration

■ Example 3 (cont'd)

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1}^{9-1} (x_{i+1} - x_i) \frac{f(x_{i+1}) + f(x_i)}{2}$$

$$= (1.25 - 1)\frac{(0.109722 + 0.145310)}{2} + (1.5 - 1.25)\frac{(0.029035 + 0.109722)}{2}$$

$$+ (1.75 - 1.5)\frac{(-0.080816 - 0.029035)}{2} + (2 - 1.75)\frac{(-0.198424 - 0.080816)}{2}$$

$$+ \ldots +$$

$$= 0.031879 + 0.017345 - 0.006473 - 0.034905 + \ldots +$$

$$= -\underline{0.345543}$$

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$f(x)$	0.145310	0.109722	0.029035	-0.080816	-0.198424	-0.303224	-0.379833	-0.420008	-0.422561

Numerical Integration

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■ Example 3 (cont'd)

x	$f(x)$	Trapezidal Rule
1.00	0.145310	
1.25	0.109722	0.031879
1.50	0.029035	0.017345
1.75	-0.080816	-0.006473
2.00	-0.198424	-0.034905
2.25	-0.303224	-0.062706
2.50	-0.379833	-0.085382
2.75	-0.420008	-0.099980
3.00	-0.422561	-0.105321
	Σ	**-0.345543**

Numerical Integration

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■ Example 3 (cont'd)

Comparison:

	Trapezoidal Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.337049	-0.343924	-0.345543	-0.346078
% error	2.61	0.62	0.15	0.0

Numerical Integration

■ Simpson's Rules

– It was noticed that the trapezoidal rule is based on a linear interpolating polynomial of the type

$$f(x) = b_0 + b_1 x \qquad (3)$$

– The accuracy in an estimate of an integral can usually be improved by using a higher-order polynomial as the interpolation formula.

Numerical Integration

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■ Simpson's Rules

– Simpson's rules are numerical formulas for estimating the value of an integral when a *second-order polynomial* or *third-order polynomial* is used as the interpolating formula, that is

$$f(x) = b_0 + b_1 x + b_2 x^2 \qquad (4)$$

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_2 x^3 \qquad (5)$$

Numerical Integration

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- Simpson's Rules
  - A Simpson's rule that is based on a second-order polynomial is referred to as the *Simpson's 1/3 Rule.*
  - A Simpson's rule that is based on a third-order polynomial is referred to as the *Simpson's 1/8 Rule.*

Numerical Integration

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- Simpson's 1/3 Rule
  - Derivation
    - Simpson's 1/3 method of integration is more accurate than the trapezoidal rule in that it assumes a second-order interpolating polynomial to approximate a given function.
    - The value of the integral for $x = -\Delta x$ to $x = \Delta x$,as shown in the figure, is approximated as

$$I \approx \int_{-\Delta x}^{\Delta x} f(x)dx \qquad (6)$$

Numerical Integration

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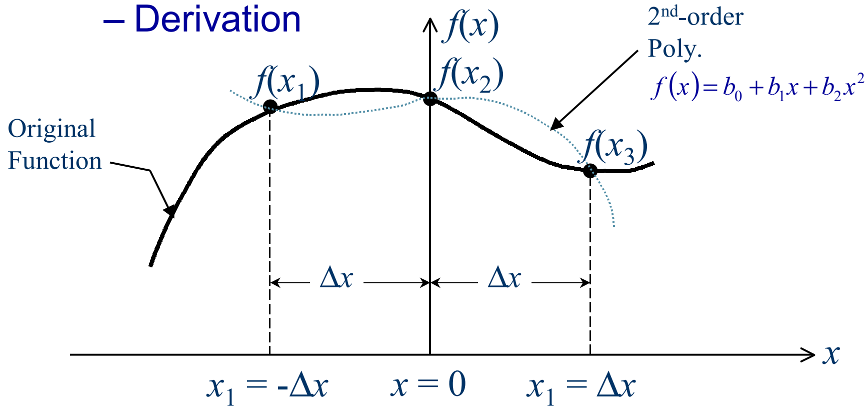
- Simpson's 1/3 Rule
  - Derivation



Numerical Integration

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- Simpson's 1/3 Rule
  - Derivation

    The three data points considered in the previous figure are tabulated as follows

Table 1

i	1	2	3
x	x_1	x_2	x_3
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Numerical Integration

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■ Simpson's 1/3 Rule

– Derivation

Where $f(x) = b_0 + b_1 x + b_2 x^2$

Integrating the right-hand side of Eq. 4, gives

$$I = \int_{-\Delta x}^{\Delta x} \left(b_0 + b_1 x + b_2 x^2\right) dx$$

$$= b_0 x + \frac{b_1 x^2}{2} + \frac{b_2 x^3}{3}\Bigg|_{-\Delta x}^{\Delta x} \qquad (7)$$

$$= 2\Delta x b_0 + \frac{2\Delta x}{3} b_2$$

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■ Simpson's 1/3 Rule

– Derivation

Eq. 7 can be rewritten in matrix form as

$$I = \begin{bmatrix} 2\Delta x & 0 & \dfrac{2\Delta x}{3} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \Delta x \\ b_2 (\Delta x)^2 \end{bmatrix} \qquad (8)$$

Or in compact matrix form

$$I = [k][b] \qquad (9)$$

Numerical Integration

■ Simpson's 1/3 Rule

– Derivation

The vector [b] can be determined by using the three data points of Table 1 (also shown in the figure). That is,

$$\begin{aligned} f(x_1) &= f(-\Delta x) = b_0 - \Delta x b_1 + (\Delta x)^2 b_2 \\ f(x_2) &= f(0) = b_0 + (0)b_1 + (0)^2 b_2 \\ f(x_3) &= f(\Delta x) = b_0 + \Delta x b_1 + (\Delta x)^2 b_2 \end{aligned} \qquad (10)$$

Numerical Integration

■ Simpson's 1/3 Rule

– Derivation

Eq 10 can easily be put in matrix form to give

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \Delta x \\ b_2 \Delta x \end{bmatrix} \qquad (11)$$

Numerical Integration

■ Simpson's 1/3 Rule

– Derivation

Solving Eq. 11 for the vector [*b*], results in

$$\begin{bmatrix} b_0 \\ b_1\Delta h \\ b_2(\Delta x)^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

or

$$\begin{bmatrix} b_0 \\ b_1\Delta h \\ b_2(\Delta x)^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \qquad (12)$$

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■ Simpson's 1/3 Rule

– Derivation

Substituting Eq. 12 into Eq. 8, the following result can be obtained:

$$I = \begin{bmatrix} 2\Delta x & 0 & \frac{2\Delta x}{3} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1\Delta x \\ b_2(\Delta x)^2 \end{bmatrix}$$

$$I = \begin{bmatrix} 2\Delta x & 0 & \frac{2\Delta x}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \qquad (13)$$

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- Simpson's 1/3 Rule
  - Derivation

    Performing the matrix multiplication on Eq. 13, yields Simpson's 1/3 Rule as follows:

$$I = \begin{bmatrix} \frac{\Delta x}{3} & \frac{4\Delta x}{3} & \frac{\Delta x}{3} \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

0r

$$I = \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + f(x_3)\right] \qquad (14)$$

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- Simpson's 1/3 Rule
  - Derivation

    Equation 14 is the basic Simpson's 1/3 rule formula, which is based on second-order interpolating polynomial. Changing the notation for x_1, x_2, and x_3 to x_i, x_{i+1}, and x_{i+2}, respectively, Eq. 14 becomes

$$I = \frac{\Delta x}{3}\left[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right] \qquad (15)$$

Numerical Integration

■ Simpson's 1/3 Rule

For $n - 1$ intervals of equal size, the Simpson's 1/3 Rule can be expressed as

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,3,5,\ldots}^{n-2} \frac{x_{i+1} - x_i}{3}\left[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right] \quad (16)$$

Note: Simpson's 1/3 Rule can only be applied when there are an even number of subintervals, or an odd number of data pairs x and $f(x)$.

Numerical Integration

■ Simpson's 3/8 Rule

– In a similar manner to the derivation of the Simpson's 1/3 Rule, a third-order polynomial (Eq. 5) can be fit to four points and integrated to yield

$$I = \frac{3\Delta x}{8}\left[f(x_1) + 3f(x_2) + 3f(x_3) + f(x_4)\right] \quad (17)$$

Numerical Integration

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■ Simpson's 3/8 Rule

For n intervals of equal size, the Simpson's 3/6 Rule can be expressed as

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,4,7,\ldots}^{n-3} \frac{3(x_{i+1}-x_i)}{8}\left[f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right] \quad (18)$$

Note: Simpson's 3/8 Rule can only be applied when there are an odd number of subintervals.

Numerical Integration

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■ Example 4 – Simpson's 1/3 Rule

Evaluate the following integral using the Simpson's 1/3 rule. Use interval widths of 1, 0.5, and 0.25, and compare your results with the true value of I = -0.346078:

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx$$

Numerical Integration

■ Example 4 (cont'd) -Simpson's 1/3 Rule

For interval width of 1, $n = \frac{3-1}{1} + 1 = 2 + 1 = 3$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	2	3
$f(x)$	0.145310	-0.198424	-0.422561

Numerical Integration

■ Example 4 (cont'd) - Simpson's 1/3 Rule

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,3,5,\ldots}^{n-2} \frac{x_{i+1}-x_i}{3}\left[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})\right]$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1}^{3-2} \frac{(x_2 - x_1)}{3}\left[f(x_1) + 4f(x_2) + f(x_3)\right]$$

$$= \frac{(2-1)}{3}\left[0.145310 + 4(-0.198424) - 0.422561\right]$$

$$= \underline{-0.356982}$$

x	1	2	3
$f(x)$	0.145310	-0.198424	-0.422561

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

For interval width of 0.5, $n = \frac{3-1}{0.5} + 1 = 4 + 1 = 5$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	1.5	2	2.5	3
$f(x)$	0.145310	0.029035	-0.198424	-0.379833	-0.422561

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,3,5,\ldots}^{n-2} \frac{x_{i+1}-x_i}{3}\left[f(x_i)+4f(x_{i+1})+f(x_{i+2})\right]$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1,3}^{5-2} \frac{(x_2-x_1)}{3}\left[f(x_1)+4f(x_2)+f(x_3)+f(x_3)+4f(x_4)+f(x_5)\right]$$

$$= \frac{(1.5\text{-}1)}{3}\begin{bmatrix} 0.145310 + 4(0.029035) - 0.198424 \\ -0.198424 + 4(-0.379833) - 0.422561 \end{bmatrix}$$

$$= \underline{-0.346215}$$

x	1	1.5	2	2.5	3
$f(x)$	0.145310	0.029035	-0.198424	-0.379833	-0.422561

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

For interval width of 0.25, $n = \frac{3-1}{0.25} + 1 = 8 + 1 = 9$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$f(x)$	0.145310	0.109722	0.029035	-0.080816	-0.198424	-0.303224	-0.379833	-0.420008	-0.422561

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,3,5,\ldots}^{n-2} \frac{x_{i+1}-x_i}{3}[f(x_i)+4f(x_{i+1})+f(x_{i+2})]$$

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx \approx \sum_{i=1,3,5,7}^{9-2} \frac{(x_2-x_1)}{3}\begin{bmatrix} f(x_1)+4f(x_2)+f(x_3)+f(x_3)+4f(x_4)+f(x_5) \\ f(x_5)+4f(x_6)+f(x_7)+f(x_7)+4f(x_8)+f(x_9) \end{bmatrix}$$

$$= \frac{(1.25-1)}{3}\begin{bmatrix} 0.145310+4(0.109722)+0.029035 \\ +0.029035+4(-0.080816)-0.198424 \\ -0.198424+4(-0.303224)-0.379833 \\ -0.379833+4(-0.420008)-0.422561 \end{bmatrix}$$

$$= \underline{-0.346083}$$

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
$f(x)$	0.145310	0.109722	0.029035	-0.080816	-0.198424	-0.303224	-0.379833	-0.420008	-0.422561

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

Comparison:

	Simpson's 1/3 Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.356982	-0.346215	-0.346083	-0.346078
% error	3.151	0.040	0.001	0.0

Numerical Integration

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■ Example 4 (cont'd) - Simpson's 1/3 Rule

– Comparison

Example 3 →

	Trapezoidal Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.337049	-0.343924	-0.345543	-0.346078
% error	2.61	0.62	0.15	0.0

	Simpson's 1/3 Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.356982	-0.346215	-0.346083	-0.346078
% error	3.151	0.040	0.001	0.0

Numerical Integration

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■ Example 5 – Simpson's 3/8 Rule

Repeat Example 4 for evaluating the following integral using the Simpson's 3/8 rule. In this case, use 3 and 6 equal size intervals, and compare your results with the true value of *I* = -0.346078:

$$\int_1^3 \frac{\cos x}{1+e^x} x^2 dx$$

Numerical Integration

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■ Example 5 (cont'd) -Simpson's 3/8 Rule

For 3 intervals of width equal to,

$n = 4$

$$w = \frac{3-1}{3} = 0.6667$$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x} x^2$$

x	1	1.6667	2.3333	3
$f(x)$	0.145310	-0.042243	-0.332453	-0.422561

Numerical Integration

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■ Example 5 (cont'd) - Simpson's 3/8 Rule

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,4,7,...}^{n-3} \frac{3(x_{i+1}-x_i)}{8}\left[f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right]$$

$$\int_1^3 \frac{\cos x}{1+e^x}x^2dx \approx \sum_{i=1}^{4-3}\frac{3(x_2-x_1)}{8}\left[f(x_1)+3f(x_2)+3f(x_3)+f(x_4)\right]$$

$$=\frac{3(1.6667-1)}{8}\left[0.14531+3(-0.042243)+3(-0.332453)-0.422561\right]$$

$$= \underline{-0.350335}$$

x	1	1.6667	2.3333	3
$f(x)$	0.145310	-0.042243	-0.332453	-0.422561

Numerical Integration

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■ Example 5 (cont'd) -Simpson's 3/8 Rule

For 6 intervals of width equal to,

$n = 7$

$$w = \frac{3-1}{6} = 0.33333$$

The following table can be constructed:

$$f(x) = \frac{\cos x}{1+e^x}x^2$$

x	1	1.33333	1.666667	2	2.333333	2.666667	3
$f(x)$	0.145310	0.087240	-0.042243	-0.198424	-0.332453	-0.410872	-0.422561

Numerical Integration

■ Example 5 (cont'd) - Simpson's 3/8 Rule

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1,4,7,\ldots}^{n-3} \frac{3(x_{i+1}-x_i)}{8}\left[f(x_i)+3f(x_{i+1})+3f(x_{i+2})+f(x_{i+3})\right]$$

$$\int_1^3 \frac{\cos x}{1+e^x}x^2dx \approx \sum_{i=1,4}^{7-3}\frac{3(x_2-x_1)}{8}\begin{bmatrix} f(x_1)+3f(x_2)+3f(x_3)+f(x_4) \\ f(x_4)+3f(x_5)+3f(x_6)+f(x_7)\end{bmatrix}$$

$$=\frac{3(1.3333-1)}{8}\begin{bmatrix}0.14531+3(0.08724)+3(-0.042243)-0.198424 \\ -0.198424+3(-0.332453)+3(-0.410872)-0.422561\end{bmatrix}$$

$$=\underline{-0.346135}$$

x	1	1.33333	1.666667	2	2.333333	2.666667	3
$f(x)$	0.145310	0.087240	-0.042243	-0.198424	-0.332453	-0.410872	-0.422561

Numerical Integration

■ Example 5 (cont'd) - Simpson's 3/8 Rule

– Comparison

Example 3 ⟹

	Trapezoidal Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.337049	-0.343924	-0.345543	-0.346078
% error	2.61	0.62	0.15	0.0

	Simpson's 3/8 Rule		True
	$n = 4$	$n = 7$	
I	-0.350335	-0.346135	-0.346078
% error	1.23	0.016	0.0

	Simpson's 1/3 Rule			True
	$n = 3$	$n = 5$	$n = 9$	
I	-0.356982	-0.346215	-0.346083	-0.346078
% error	3.151	0.040	0.001	0.0