CHAPTER 7c. DIFFERENTIATION AND INTEGRATION

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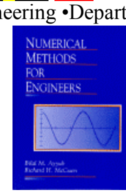

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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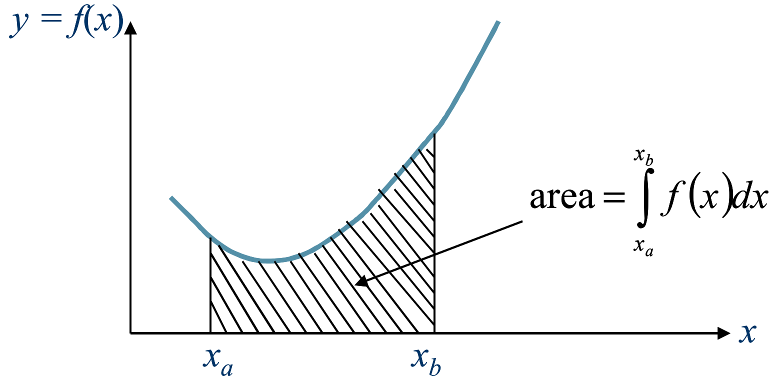
Numerical Integration

- In calculus, integration is used to find the area under the curve.
- In engineering applications, the area under the curve can have physical interpretation and implications.
- For example, it can mean finding the total energy or rate of flow Q through a cross section of a river.

Numerical Integration

■ Area Under the Curve



Numerical Integration

■ Examples

$$y = f(x) = 1 + x$$

$$I = \int_0^1 f(x)dx = \int_0^1 (1+x)dx = \left(x + \frac{x^2}{2}\right)\Bigg]_0^1 = \frac{1}{2}$$

$$y = f(x) = x^3 - e^x + \sin x$$

$$\Rightarrow \int y dx = \int f(x)dx = \int \left(x^3 - e^x + \sin x\right)dx$$

$$= \frac{x^4}{4} - e^x - \cos x + c$$

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- Need for Numerical Integration

  In general, the function to be integrated will typically be in one of the following forms:

  1. A simple continuous linear function such as a polynomial, an exponential, or trigonometric function, such as

$$f(x) = 3x^5 - 2x + 20$$
$$f(x) = 1 - e^{2x}$$
$$f(x) = 32 + \sin x$$

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- Need for Numerical Integration

  2. A complex non-linear continuous function that is difficult or impossible to integrate directly such as

$$f(x) = \frac{1 + x^3 + \sin x^2}{1 + \cos x}$$
$$f(x) = x^4 - \frac{1}{x} + x(\tan x - e^x)$$
$$f(x) = \frac{e^x}{3 + x^2} + x \ln x - \frac{\cos x}{x}$$

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3. A tabulated continuous function where values of the independent variable x and $f(x)$ are given at a number of discrete data points as is often the case with experimental or field data such as distance traveled by a car vs. time:

t (sec)	0	2	4	6	8	10
D (ft)	0	10	50	150	330	610

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■ Need for Numerical Integration

– In the first case, the integral of a simple function may be computed analytically using calculus.

– For the second case, analytical solutions are often impractical and sometimes difficult or impossible to obtain.

– In these situations as well as in the third case, approximate methods must be used.

Numerical Integration

- Need for Numerical Integration
  - Pre computers and computational devices, a visually oriented approach were used to integrate tabulated data and complicated functions.
  - In this approach, the function is plotted on a grid (see figure), and the number of boxes that approximate the area are counted.

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- Need for Numerical Integration
  - This number is multiplied by the area of each box to give a rough estimate of the total area under the curve.
  - This estimate can be refined at the expense of additional effort by using a finer grid.

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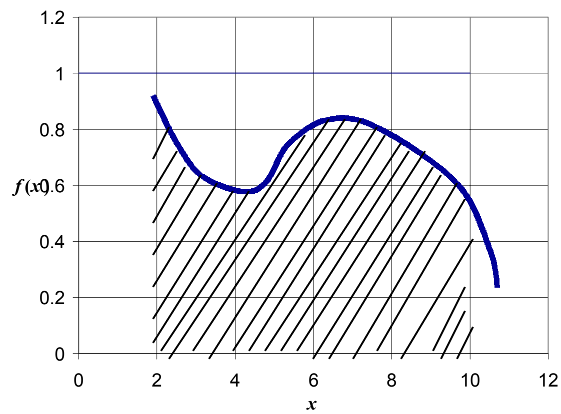
Numerical Integration

■ Need for Numerical Integration



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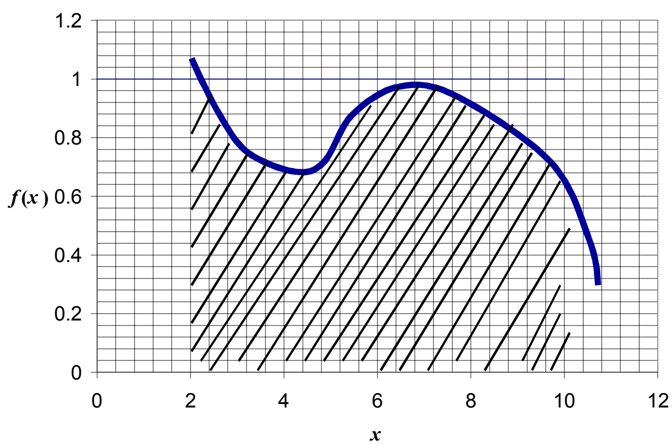
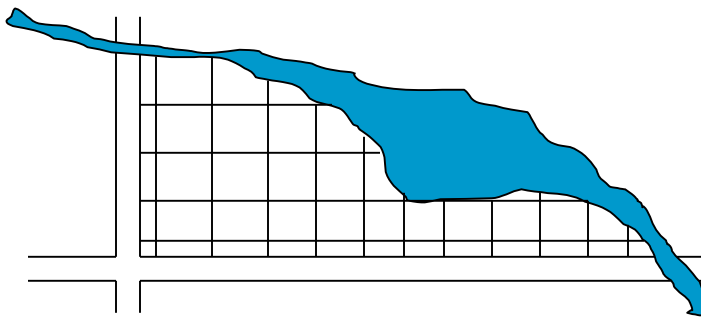
■ Need for Numerical Integration



Numerical Integration

■ Engineering Applications

- A surveyor might need to know the area of a piece of land bounded by a meandering stream and two roads



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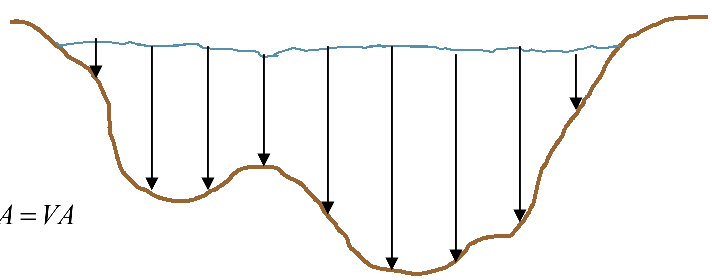
■ Engineering Applications

- A water-resource engineer might need to know the cross-sectional area of a river to calculate the rate of flow Q.

$$Q = \int V dA = VA$$



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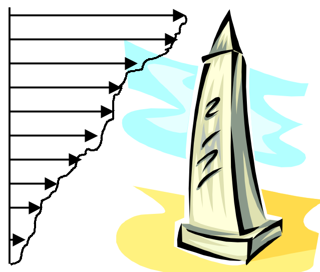
■ Engineering Applications

- A structural engineer might need to determine the net lateral force due to non-uniform wind blowing against a side of a tall building.



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■ Integration Using Interpolating Polynomial

- The general form of an interpolating polynomial is given by

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

- This polynomial can be integrated analytically as follows:

$$\int f(x)dx = b_0 x + \frac{b_1 x^2}{2} + \frac{b_2 x^3}{3} + \ldots + \frac{b_{n-1} x^n}{n} + \frac{b_n x^{n+1}}{n+1}$$

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- Integration Using Interpolating Polynomial
  - The Gregory-Newton method for deriving an interpolation formula can also be used to evaluate the integral of a function.
  - Recall G-N method:

$$f(x)=a_1+a_2(x-x_1)+a_3(x-x_1)(x-x_2)+a_4(x-x_1)(x-x_2)(x-x_3)$$
$$+a_n(x-x_1)(x-x_2)\ldots(x-x_{n-1})+a_{n+1}(x-x_1)(x-x_2)\ldots(x-x_n)$$

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- Integration Using Interpolating Polynomial
  - The terms could be rearrange to form an *n*th-order polynomial of the type

$$f(x)=b_1x^n+b_2x^{n-1}+\cdots+b_{n+1}$$

  - This polynomial can be integrated analytically as

$$\int f(x)dx=\frac{b_1}{n+1}x^{n+1}+\frac{b_2}{n}x^n+\cdots+b_{n+1}x$$

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■ Example 1

Derive a second-degree interpolation polynomial to fit the following data points, and then using the fitted polynomial to approximate $\int_{1.5}^{2.5} f(x)dx$

Compare your result with that of the exact integral (i.e., $x^4/4$).

x	1	2	3
$f(x)$	1	8	27

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■ Example 1 (cont'd)

– The general form of the interpolation polynomial is given by

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n$$

– In our case it is

$$f(x) = b_0 + b_1 x + b_2 x^2$$

Numerical Integration

- Example 1 (cont'd)
  - We need to find the coefficients b_0, b_1, and b_2.
  - We notice that have three unknowns, $n = 3$, that require solving 3 simultaneous linear equations using the pair, x_i and $f(x_i)$, of the given data as follows:

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- Example 1 (cont'd)

x	1	2	3
$f(x)$	1	8	27

$$f(x) = b_0 + b_1 x + b_2 x^2$$

$$1 = b_0 + b_1(1) + b_2(1)^2$$
$$8 = b_0 + b_1(2) + b_2(2)^2$$
$$27 = b_0 + b_1(3) + b_2(3)^2$$

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■ Example 1 (cont'd)

$$b_0 + b_1 + b_2 = 1$$
$$b_0 + 2b_1 + 4b_2 = 8$$
$$b_0 + 3b_1 + 9b_2 = 27$$

– OR

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 27 \end{bmatrix}$$

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■ Example 1 (cont'd)

– The solution of this set of equations yields

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 6 \end{bmatrix} \qquad f(x) = b_0 + b_1 x + b_2 x^2$$

– Therefore, the interpolation polynomial is

$$f(x) = 6 - 11x + 6x^2$$

– And its anti-derivative (integral) is

$$\int f(x)dx = 6x - \frac{11}{2}x^2 + \frac{6}{3}x^3 + c$$

Numerical Integration

■ Example 1 (cont'd)

The result is

$$\int_{1.5}^{2.5} f(x)dx = 6x - \frac{11}{2}x^2 + 2x^3 \Big]_{1.5}^{2.5}$$

$$= \left[6(2.5) - \frac{11}{2}(2.5)^2 + 2(2.5)^3\right] - \left[6(1.5) - \frac{11}{2}(1.5)^2 + 2(1.5)^3\right]$$

$$= 11.875 - 3.375$$

$$= 8.5$$

Numerical Integration

■ Example 1 (cont'd)

– Evaluation of the exact integral is as follows:

$$\int_{1.5}^{2.5} f(x)dx = \int_{1.5}^{2.5} x^3 = \frac{x^4}{4}\Big|_{1.5}^{2.5} = \frac{(2.5)^4 - (1.5)^4}{4} = 8.5$$

– In this example, the approximation is identical to the true value.

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- The Trapezoidal Rule
  - The trapezoidal rule approximates the area of a function defined by a set of discrete points by fitting a trapezoid to each pair of adjacent points that defines the dependent variable and summing the individual areas as shown in the figure.

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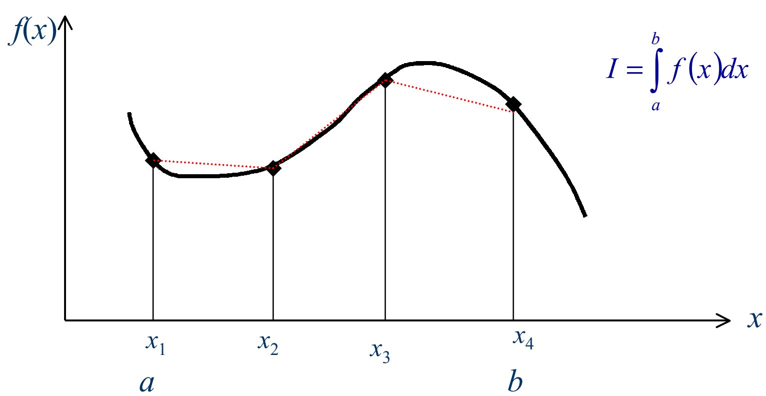
- The Trapezoidal Rule



$$I = \int_a^b f(x)dx$$

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■ The Trapezoidal Rule

– Derivation

- The trapezoidal rule can be derived by fitting a linear interpolating polynomial to each pair of points.
- Using, for example Gregory-Newton formula, an expression for a linear polynomial can be obtained as follows:

$$f(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + a_4(x - x_1)(x - x_2)(x - x_3) + a_n(x - x_1)(x - x_2)\ldots(x - x_{n-1}) + a_{n+1}(x - x_1)(x - x_2)\ldots(x - x_n)$$

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■ The Trapezoidal Rule

– Derivation

- If we denote the values of the two independent variables as x_i and x_{i+1}, then G-N formula gives

x	$f(x)$
x_i	$f(x_i)$
x_{i+1}	$f(x_{i+1})$

$$f(x) = a_1 + a_2(x - x_i)$$
$$f(x_i) = a_1 + a_2(x_i - x_i)$$
$$f(x_i) = a_1$$

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- The Trapezoidal Rule
  - Derivation

x	$f(x)$
x_i	$f(x_i)$
x_{i+1}	$f(x_{i+1})$

$$f(x) = a_1 + a_2(x - x_i)$$

$$f(x_{i+1}) = a_1 + a_2(x_{i+1} - x_i) = f(x_i) + a_2(x_{i+1} - x_i)$$

Therefore,

$$a_2 = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

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- The Trapezoidal Rule
  - Derivation
    - Hence, the linear polynomial is given by

x	$f(x)$
x_i	$f(x_i)$
x_{i+1}	$f(x_{i+1})$

$$f(x) = a_1 + a_2(x - x_i)$$

$$f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) \qquad (1)$$

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- The Trapezoidal Rule
  - Derivation
    - Integrating Eq. 1 between two points, say a and b:

$$\int_a^b f(x)dx = f(x_i)x + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\frac{(x-x_i)^2}{2}\Bigg|_a^b$$

$$= f(x_i)b + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\frac{(b-x_i)^2}{2}$$

$$- f(x_i)a - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\frac{(a-x_i)^2}{2}$$

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- The Trapezoidal Rule
  - Derivation

Or

$$\int_a^b f(x)dx = \frac{b-a}{x_{i+1} - x_i}\left[\frac{(b+a)}{2}\left(f(x_{i+1}) - f(x_i)\right) - x_{i+1}f(x_i) - x_i f(x_{i+1})\right]$$

Letting $b = x_{i+1}$ and $a = x_i$, results in

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{x_{i+1} - x_i}{2}\left[f(x_{i+1}) + f(x_i)\right]$$

Numerical Integration

■ The Trapezoidal Rule

The trapezoidal rule can be used to approximate the integral between two points x_1 and x_n of a function. It is given by

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i)\frac{f(x_{i+1}) + f(x_i)}{2} \qquad (2)$$

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■ Geometric Interpretation the Trapezoidal Rule

– The trapezoidal rule can also be derived geometrically.

– The trapezoidal rule is equivalent to approximating the area of the trapezoid under the straight line connecting $f(x_i)$ and $f(x_{i+1})$ as shown in the following figure:

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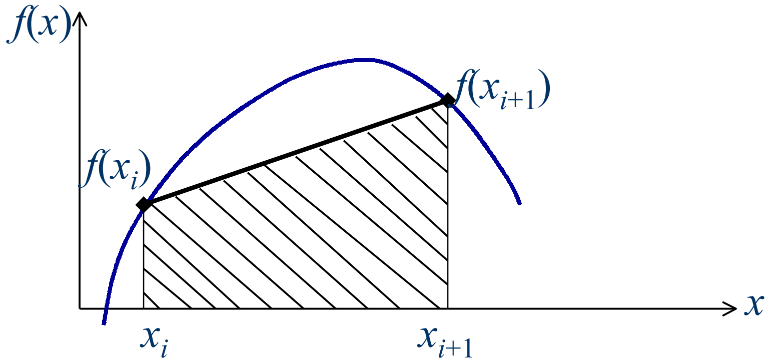
- Geometric Interpretation the Trapezoidal Rule



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- Geometric Interpretation the Trapezoidal Rule
  - Recall from geometry that the formula for computing the area of a trapezoid is the height times the average of the bases as shown in the figure.
  - Therefore, the integral estimate can be represented as

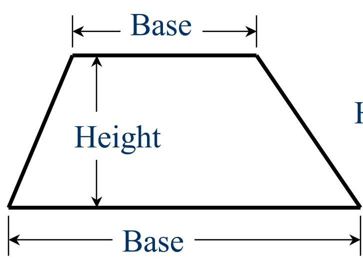
$$I \approx \text{width} \times \text{average height}$$

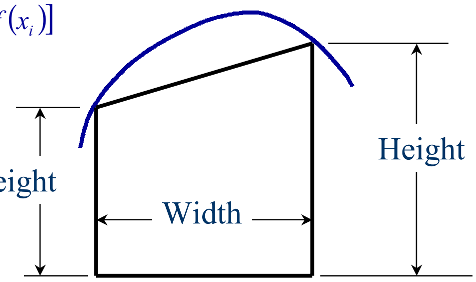
Numerical Integration

■ Geometric Interpretation the Trapezoidal Rule

$I \approx \text{width} \times \text{average height}$

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{x_{i+1} - x_i}{2}\left[f(x_{i+1}) + f(x_i)\right]$$





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■ Example 2

Using the trapezoidal rule , estimate the area under the curve, that is

$$I = \int_1^4 f(x)dx$$

for the following function given in tabulated form:

i	1	2	3	4
x_i	1	2	3	4
$f(x_i)$	3	-1	-3	3

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Numerical Integration

■ Example 2 (cont'd)

Using Eq. 2,

$$\int_{x_1}^{x_n} f(x)dx \approx \sum_{i=1}^{n-1} (x_{i+1} - x_i)\frac{f(x_{i+1}) + f(x_i)}{2}$$

$$\int_{1}^{4} f(x)dx \approx (2-1)\frac{[3+(-1)]}{2} + (3-2)\frac{[-1+(-3)]}{2} + (4-3)\frac{[3+(-3)]}{2}$$

$$= 1 + (\text{-}2) + 0$$

$$= -1$$

i	1	2	3	4
x_i	1	2	3	4
$f(x_i)$	3	-1	-3	3