CHAPTER 6f. NUMERICAL INTERPOLATION

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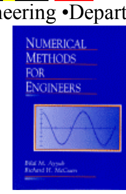

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

University of Maryland, College Park

Interpolation Using Splines

- Generally, the higher-order polynomial gives higher accuracy.
- However, this is not true in some situations, especially when the data points include local abrupt changes in $f(x)$ values for steady changes in x values.
- In these situations, the accuracy decreases.

Interpolation Using Splines

- In these cases, the function can lead to erroneous results because of round-off error and overshoot.
- An alternative approach is to use a lower-order polynomial to subsets of the data points.
- Such connecting polynomials are called *splines functions*.

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- The concept of spline came from the drafting technique of using thin flexible strip, called *spline*, to draw smooth curves through a set of points as shown in the figure of the viewgraph.
- The drafter places paper over wooden board and hammer nails into the paper and the board at the locations of the data points.

Interpolation Using Splines

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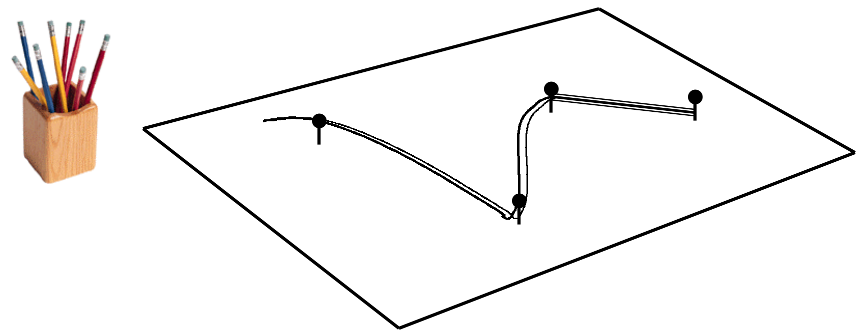
■ Origin of Spline Concept



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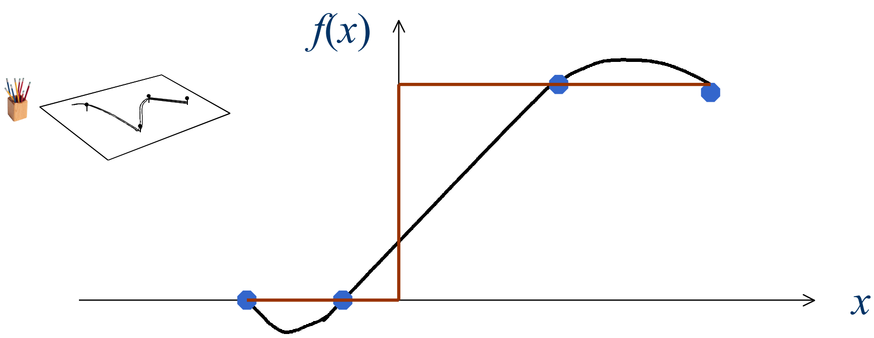
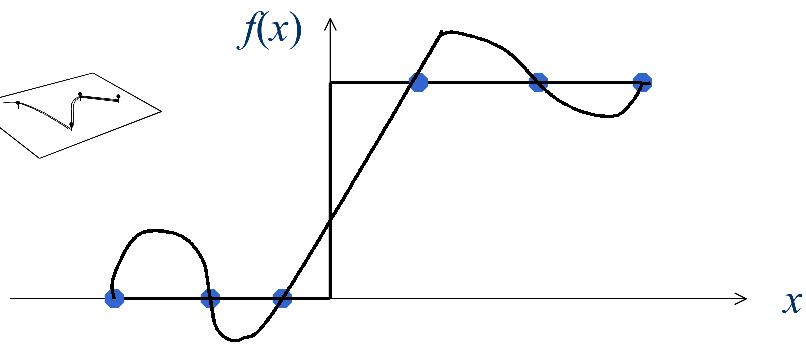
■ Need for Splines



Abrupt change indicates oscillation in interpolating polynomial.

Interpolation Using Splines

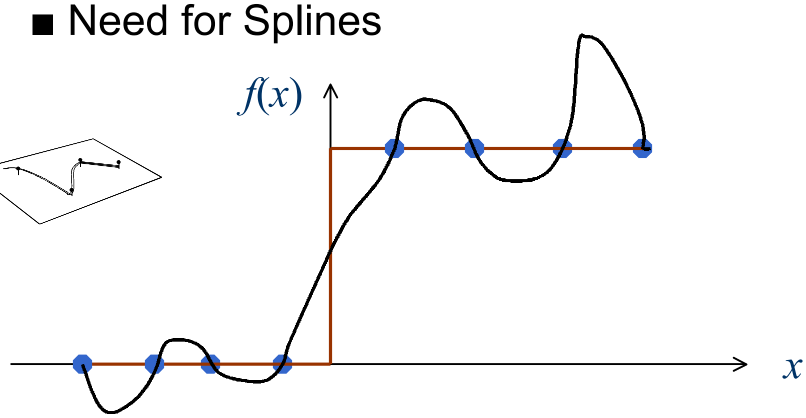
- Need for Splines



Abrupt change indicates oscillation in interpolating polynomial.

Interpolation Using Splines

- Need for Splines



Abrupt change indicates oscillation in interpolating polynomial.

Interpolation Using Splines

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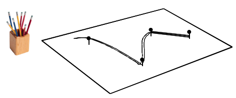
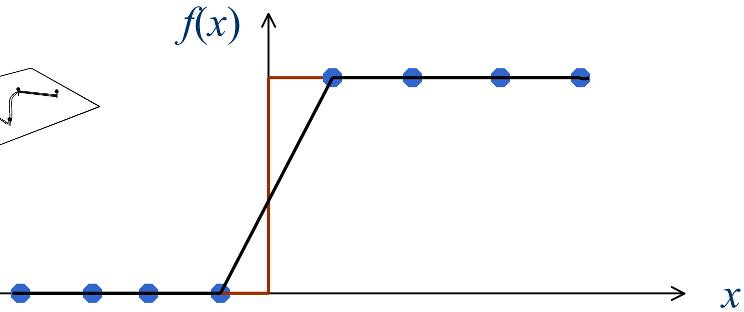
- Need for Splines





The spline provides a much more acceptable approximation.

Interpolation Using Splines

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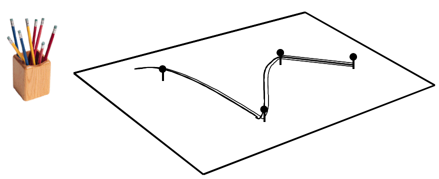
- Types of Splines
  1. Linear (simplest),
  2. Quadratic, and
  3. Cubic (most popular)



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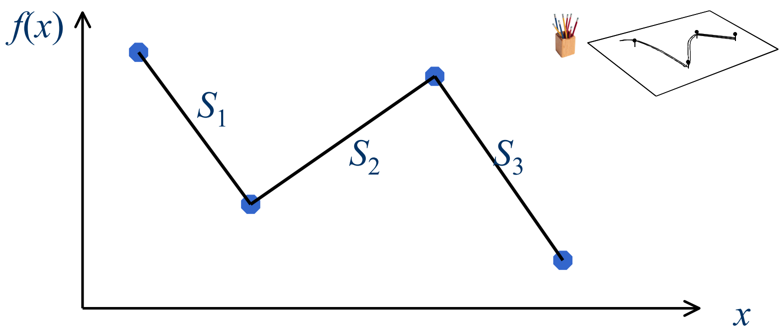
■ Linear or First-order Splines



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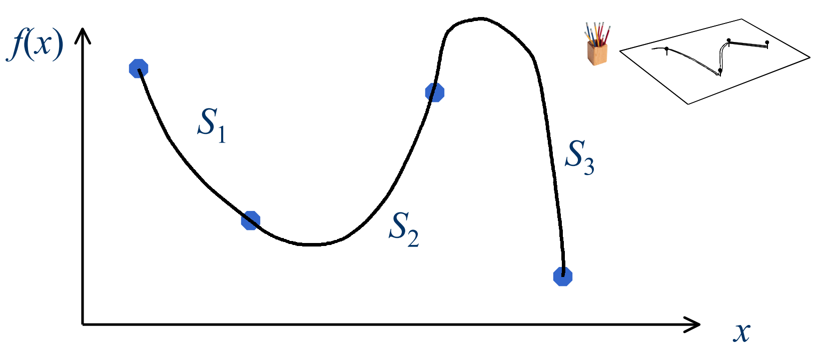
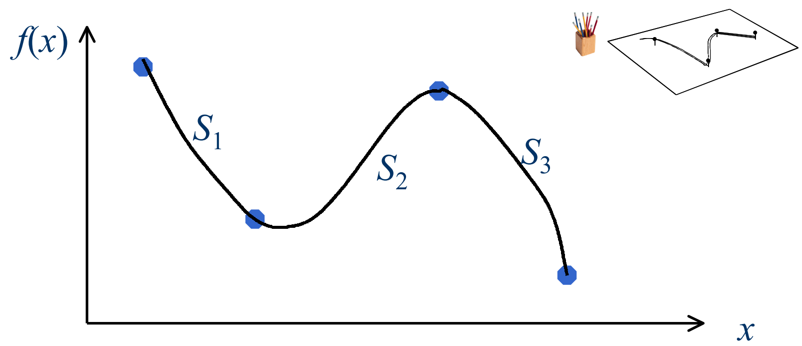
■ Quadratic or Second-order Splines



Interpolation Using Splines

■ Cubic or Third-order Splines



Interpolation Using Splines

■ Linear Splines

- Linear or first-order splines are the simplest to construct.
- They are straight lines connecting two pair of the data points.
- Therefore, an algorithm for straight lines that connect each pair of the data points can be developed very easily.

Interpolation Using Splines

■ Linear Splines

– The interpolation function can be expressed as

$$
\begin{aligned}
f_1(x) &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \qquad \text{for } x_1 \le x \le x_2 \\
f_2(x) &= f(x_2) + \frac{f(x_3) - f(x_2)}{x_3 - x_2}(x - x_2) \qquad \text{for } x_2 \le x \le x_3 \\
&\vdots \\
f_i(x) &= f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) \qquad \text{for } x_i \le x \le x_{i+1} \\
f_{n-1}(x) &= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}) \qquad \text{for } x_{n-1} \le x \le x_n
\end{aligned} \qquad (1)
$$

Interpolation Using Splines

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■ Linear Splines

– Requirements for Linear Splines:

- The linear spline functions as presented by the previous equations are to satisfy the following condition:

$$f_i(x_i) = f_{i+1}(x_i) \qquad \text{For } i = 1, 2, \ldots, n-1 \qquad (2)$$

Interpolation Using Splines

- Example 1 (Linear Splines)

  Fit the following set of data with first-order (linear) splines. Evaluate the function at $x = 5$. Also plot the spline functions.

x	3	4.5	7	9
$f(x)$	2.5	1	2.5	0.5

Interpolation Using Splines

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- Example 1 - Linear Splines
  - For the first pairs of data:

x	$f(x)$
3	2.5
4.5	1

$$f_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

$$f_1(x) = 2.5 + \frac{1 - 2.5}{4.5 - 3}(x - 3)$$

$$f_1(x) = 2.5 - (x - 3)$$

$$f(x) = 5.5 - x \qquad \text{For } 3 \le x \le 4.5$$

Interpolation Using Splines

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- Example 1 (cont'd) - Linear Splines
  - For the second pairs of data:

x	$f(x)$
4.5	1
7	2.5

$$f_2(x) = f(x_2) + \frac{f(x_3) - f(x_2)}{x_3 - x_2}(x - x_2)$$

$$f_2(x) = 1 + \frac{2.5 - 1}{7 - 4.5}(x - 4.5)$$

$$f_1(x) = 1 + 0.6(x - 4.5)$$

$$f(x) = -1.7 + 0.6x \qquad \text{For } 4.5 \le x \le 7$$

Interpolation Using Splines

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- Example 1 (cont'd) - Linear Splines
  - For the third pairs of data:

x	$f(x)$
7	2.5
9	0.5

$$f_3(x) = f(x_3) + \frac{f(x_4) - f(x_3)}{x_4 - x_3}(x - x_3)$$

$$f_2(x) = 2.5 + \frac{0.5 - 2.5}{9 - 7}(x - 7)$$

$$f_1(x) = 2.5 - (x - 7)$$

$$f(x) = 9.5 - x \qquad \text{For } 7 \le x \le 9$$

Interpolation Using Splines

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- Example 1 (cont'd) - Linear Splines
  - Finding the value of the function when $x = 5$

    If $x = 5$, then the following spline applies:

$$f(x) = -1.7 + 0.6x \qquad \text{For } 4.5 \le x \le 7$$

Thus,

$$\begin{aligned} f(5) &= -1.7 + 0.6x \\ &= -1.7 + 0.6(5) \\ &= 1.3 \end{aligned}$$

Interpolation Using Splines

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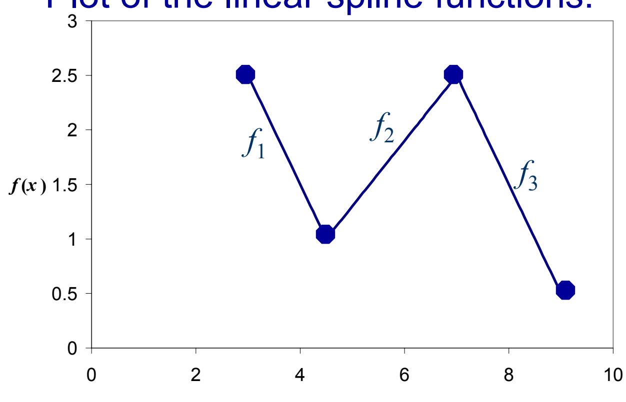
- Example 1 (cont'd) - Linear Splines

  Plot of the linear spline functions:



Interpolation Using Splines

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- Quadratic Splines

The quadratic splines provide a quadratic equation connecting any two adjacent data points within the data set of the type

$$[x_i, f(x_i)] \quad \text{for } i = 1,2,3,\ldots,n$$

i	1	2	3	4	…	n
x	x_1	x_2	x_3	x_4	…	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	…	$f(x_n)$

Interpolation Using Splines

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- Quadratic Splines

The general form of a quadratic equation between point $[x_i, f(x_i)]$ and $[x_{i+1}, f(x_{i+1})]$ is

$$f_i(x) = a_i x^2 + b_i x + c_i \qquad \text{For } i = 1,2,\ldots,n-1 \tag{3}$$

Interpolation Using Splines

- Quadratic Splines
  - Every two adjacent data points have an interpolation equation given by Eq. 3, with three constants a_i, b_i, and c_i.
  - As a result, there are 3(n – 1)unknowns that need to be determined using the data set, requiring 3(n – 1) conditions

Interpolation Using Splines

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- Quadratic Splines

  Conditions for Quadratic Splines:

  1. The splines must pas through the data points. For the *i*th spline, this condition can be expressed as

$$f_i(x_i) = a_i x_i^2 + b_i x_i + c_i = f(x_i) \qquad \text{for } i = 1,2,3,\ldots,n-1 \qquad (4a)$$

$$f_i(x_{i+1}) = a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f(x_{i+1}) \qquad \text{for } i = 1,2,3,\ldots,n-1 \qquad (4b)$$

Interpolation Using Splines

- Quadratic Splines

Conditions for Quadratic Splines:

2. The splines must be continuous at the interior data points. This condition can be expressed using the first derivatives of the quadratic splines as

$$2a_i x_{i+1} + b_i = 2a_{i+1} x_{i+1} + b_{i+1} \qquad \text{for } i = 1,2,3,\ldots,n-2 \qquad (5)$$

Interpolation Using Splines

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- Quadratic Splines

Conditions for Quadratic Splines:

3. The last condition can be arbitrary. The second derivative for the spline between the first two data points can be set equal to zero. Since the second derivative for the first spline is $2a_1$, this condition can be written as

$$2a_1 = 0 \Rightarrow a_1 = 0 \qquad (6)$$

Interpolation Using Splines

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- Quadratic Splines
  – Equations 4 to 6 provide the needed $3(n-1)$ conditions to solve for the $3(n-1)$ unknowns a_i, b_i, and c_i, $i = 1,2,\ldots, n-1$.
  – The resulting quadratic splines provide the desired continuity while passing through the data points.

ENCE 203 – CHAPTER 6f. NUMERICAL INTERPOLATION

Slide No. 191

Interpolation Using Splines

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- Example 2 – Quadratic Splines

  Fit the following set of data with second-order (quadratic) splines. Evaluate the function at $x = 5$. Also plot the spline functions.

i	1	2	3	4
x	3	4.5	7	9
$f(x)$	2.5	1	2.5	0.5

ENCE 203 – CHAPTER 6f. NUMERICAL INTERPOLATION

Slide No. 192

Interpolation Using Splines

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■ Example 2 (cont'd) – Quadratic Splines

– For the first data pairs, Eqs. 4a and 4b apply as follows:

$$a_i x_i^2 + b_i x_i + c_i = f(x_i) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

$$a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f(x_{i+1}) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

i	x	$f(x)$
1	3	2.5
2	4.5	1

$$a_1(3)^2 + b_1(3) + c_1 = 2.5$$
$$a_1(4.5)^2 + b_1(4.5) + c_1 = 1 \qquad (7)$$

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Slide No. 193

Interpolation Using Splines

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■ Example 2 (cont'd) – Quadratic Splines

– For the second data pairs, Eqs. 4a and 4b apply as follows:

$$a_i x_i^2 + b_i x_i + c_i = f(x_i) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

$$a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f(x_{i+1}) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

i	x	$f(x)$
2	4.5	1
3	7	2.5

$$a_2(4.5)^2 + b_2(4.5) + c_2 = 1$$
$$a_2(7)^2 + b_2(7) + c_2 = 2.5 \qquad (8)$$

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Slide No. 194

Interpolation Using Splines

- Example 2 (cont'd) – Quadratic Splines
  - For the third data pairs, Eqs. 4a and 4b apply as follows:

$$a_i x_i^2 + b_i x_i + c_i = f(x_i) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

$$a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f(x_{i+1}) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

i	x	$f(x)$
3	7	2.5
4	9	0.5

$$a_3(7)^2 + b_3(7) + c_3 = 2.5$$
$$a_3(9)^2 + b_3(9) + c_3 = 0.5 \qquad (9)$$

Interpolation Using Splines

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- Example 2 (cont'd) – Quadratic Splines
  - For the first and second data pairs, Eqs. 5 applies as follows:

$$2a_i x_{i+1} + b_i = 2a_{i+1} x_{i+1} + b_{i+1} \qquad \text{for } i = 1,2,3,\ldots,n-2$$

i	x	$f(x)$
2	4.5	1
3	7	2.5

$$2a_1(4.5) + b_1 = 2a_2(4.5) + b_2$$
$$2a_2(7) + b_2 = 2a_3(7) + b_3 \qquad (10)$$

Interpolation Using Splines

■ Example 2 (cont'd) – Quadratic Splines

– The last condition comes from Eq. 6 as

$$a_1 = 0$$

– Since $a_1 = 0$, the resulting system of 3(*n*-1) - 1= 3(4 – 1)-1 = 8 equations can be obtained as follows:

Interpolation Using Splines

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■ Example 2 (cont'd) – Quadratic Splines

$$
\begin{aligned}
3b_1 + c_1 &= 2.5 \\
4.5b_1 + c_1 &= 1 \\
20.25a_2 + 4.5b_2 + c_2 &= 1 \\
49a_2 + 7b_2 + c_2 &= 2.5 \\
49a_3 + 7b_3 + c_3 &= 2.5 \\
81a_3 + 9b_3 + c_3 &= 0.5 \\
b_1 - 9a_2 - b_2 &= 0 \\
14a_2 + b_2 - 14a_3 - b_3 &= 0
\end{aligned}
\qquad (11)
$$

Interpolation Using Splines

■ Example 2 (cont'd) – Quadratic Splines

Eq. 11 can be rewritten in a matrix form as

$$\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\ 1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & 1 & 0 & -14 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

Interpolation Using Splines

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■ Example 2 (cont'd) – Quadratic Splines

– Therefore,

$$\begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5.5 \\ 0.64 \\ -6.76 \\ 18.46 \\ -1.6 \\ 24.6 \\ -91.3 \end{bmatrix}$$

$$f_1(x) = x + 5.5 \quad \text{for } 3 \le x \le 4.5 \quad (13a)$$

$$f_2(x) = 0.64x^2 - 6.76x + 18.46 \quad \text{for } 4.5 \le x \le 7 \quad (13b)$$

$$f_3(x) = -1.6x^2 + 24.6x - 91.3 \quad \text{for } 7 \le x \le 9 \quad (13c)$$

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Interpolation Using Splines

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- Example 1 (cont'd) - Quadratic Splines
  - Finding the value of the function when $x = 5$

    If $x = 5$, then the following spline of Eq. 13b applies:

$$f_2(x) = 0.64x^2 - 6.76x + 18.46 \quad \text{for } 4.5 \le x \le 7$$

Thus,

$$f(5) = 0.64(5)^2 - 6.76(5) + 18.46$$
$$= 0.66$$

Interpolation Using Splines

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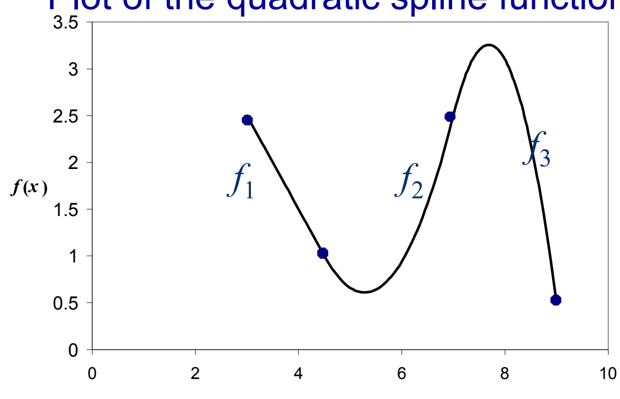
- Example 2 (cont'd) - Quadratic Splines

  Plot of the quadratic spline functions:



Interpolation Using Splines

■ Cubic Splines

The cubic splines provide a cubic equation connecting any two adjacent data points within the data set of the type

$$[x_i, f(x_i)] \quad \text{for } i = 1,2,3,\ldots,n$$

i	1	2	3	4	…	n
x	x_1	x_2	x_3	x_4	…	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	…	$f(x_n)$

Interpolation Using Splines

■ Cubic Splines

The general form of a cubic equation between point $[x_i, f(x_i)]$ and $[x_{i+1}, f(x_{i+1})]$ is

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \qquad \text{For } i = 1,2,\ldots,n-1 \qquad (14)$$

Interpolation Using Splines

- Cubic Splines
  - Every two adjacent data points have an interpolation equation given by Eq. 14, with three constants a_i, b_i, c_i, and d_i.
  - As a result, there are $4(n - 1)$ unknowns that need to be determined using the data set, requiring $4(n - 1)$ conditions

Interpolation Using Splines

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- Cubic Splines

  Conditions for Cubic Splines:

  1. The splines must pas through the data points. For the *i*th spline, this condition can be expressed as

$$f_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i = f(x_i) \quad \text{for } i = 1,2,3,\ldots,n-1 \quad (15a)$$

$$f_i(x_{i+1}) = a_i x_{i+1}^3 + b_i x_{i=1}^2 + c_i x_i + d_i = f(x_{i+1}) \quad \text{for } i = 1,2,3,\ldots,n-1 \quad (15b)$$

Interpolation Using Splines

- Cubic Splines

Conditions for Quadratic Splines:

2. The splines must be continuous at the interior data points. This condition can be expressed using the first derivatives of the cubic splines as

$$3a_i x_{i+1}^2 + 2b_i x_{i+1} + c_i = 3a_{i+1} x_{i+1}^2 + 2b_{i+1} x_{i+1} + c_{i+1} \quad (16)$$
$$\text{for } i = 1,2,3,\ldots,n-2$$

Interpolation Using Splines

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- Cubic Splines

Conditions for Quadratic Splines:

3. The splines must satisfy the second derivative continuity at the interior points. This condition can expressed as

$$6a_i x_{i+1} + 2b_i = 6a_{i+1} x_{i+1} + 2b_{i+1} \quad (17)$$
$$\text{for } i = 1,2,3,\ldots,n-2$$

Interpolation Using Splines

■ Cubic Splines

Conditions for Quadratic Splines:

4. The last condition can be set as the second derivative at the first and last data points to be zero. This condition is as follows:

$$6a_1 x_i + 2b_1 = 0$$
$$6a_{n-1} x_n + 2b_{n-1} = 0 \quad (18)$$

Interpolation Using Splines

■ Cubic Splines

– Equations 15 to 18 provide the needed $4(n - 1)$ conditions to solve for the $4(n - 1)$ unknowns a_i, b_i, c_i, and d_i, $i = 1,2,\ldots, n - 1$.

– The resulting cubic splines provide the desired continuity while passing through the data points.

Interpolation Using Splines

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■ Example 3 – Cubic Splines

Fit the following set of data with third-order (cubic) splines. Evaluate the function at $x = 5$. Also plot the spline functions.

i	1	2	3	4
x	3	4.5	7	9
$f(x)$	2.5	1	2.5	0.5

Interpolation Using Splines

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■ Example 3 (cont'd) – Cubic Splines

– For the first data pairs, Eqs. 15a and 15b apply as follows:

$$f_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i = f(x_i) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

$$f_i(x_{i+1}) = a_i x_{i+1}^3 + b_i x_{i+1}^2 + c_i x_i + d_i = f(x_{i+1}) \qquad \text{for } i = 1,2,3,\ldots,n-1$$

i	x	$f(x)$
1	3	2.5
2	4.5	1

$$a_1(3)^3 + b_1(3)^2 + c_1(3) + d_1 = 2.5$$
$$a_1(4.5)^3 + b_1(4.5)^2 + c_1(4.5) + d_1 = 1 \qquad (19)$$

Interpolation Using Splines

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- Example 3 (cont'd) – Cubic Splines
  - For the second data pairs, Eqs. 15a and 15b apply as follows:

$$f_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i = f(x_i) \quad \text{for } i = 1,2,3,\ldots,n-1$$

$$f_i(x_{i+1}) = a_i x_{i+1}^3 + b_i x_{i+1}^2 + c_i x_i + d_i = f(x_{i+1}) \quad \text{for } i = 1,2,3,\ldots,n-1$$

i	*x*	*f*(*x*)
2	4.5	1
3	7	2.5

$$a_2(4.5)^3 + b_2(4.5)^2 + c_2(4.5) + d_2 = 1$$
$$a_2(7)^3 + b_2(7)^2 + c_2(7) + d_2 = 2.5 \qquad (20)$$

ENCE 203 – CHAPTER 6f. NUMERICAL INTERPOLATION

Slide No. 213

Interpolation Using Splines

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- Example 3 (cont'd) – Cubic Splines
  - For the third data pairs, Eqs. 15a and 15b apply as follows:

$$f_i(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i = f(x_i) \quad \text{for } i = 1,2,3,\ldots,n-1$$

$$f_i(x_{i+1}) = a_i x_{i+1}^3 + b_i x_{i+1}^2 + c_i x_i + d_i = f(x_{i+1}) \quad \text{for } i = 1,2,3,\ldots,n-1$$

i	*x*	*f*(*x*)
3	7	2.5
4	9	0.5

$$a_3(7)^3 + b_3(7)^2 + c_3(7) + d_3 = 2.5$$
$$a_3(9)^3 + b_3(9)^2 + c_3(9) + d_3 = 0.5 \qquad (21)$$

ENCE 203 – CHAPTER 6f. NUMERICAL INTERPOLATION

Slide No. 214

Interpolation Using Splines

- Example 3 (cont'd) – Cubic Splines
  - For the second and third data pairs, Eq. 16 applies as follows:

i	x	$f(x)$
2	4.5	1
3	7	2.5

$$3a_i x_{i+1}^2 + 2b_i x_{i+1} + c_i = 3a_{i+1} x_{i+1}^2 + 2b_{i+1} x_{i+1} + c_{i+1}$$
$$\text{for } i = 1,2,3,\ldots,n-2$$

$$3a_1(4.5)^2 + 2b_1(4.5) + c_1 = 3a_2(4.5)^2 + 2b_2(4.5) + c_2$$
$$3a_2(7)^2 + 2b_2(7) + c_2 = 3a_3(7)^2 + 2b_3(7) + c_3 \qquad (22)$$

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- Example 3 (cont'd) – Cubic Splines
  - For the second and third data pairs, Eq. 17 applies as follows:

i	x	$f(x)$
2	4.5	1
3	7	2.5

$$6a_i x_{i+1} + 2b_i = 6a_{i+1} x_{i+1} + 2b_{i+1}$$
$$\text{for } i = 1,2,3,\ldots,n-2$$

$$6a_1(4.5) + 2b_1 = 6a_2(4.5) + 2b_2$$
$$6a_2(7) + 2b_2 = 6a_3(7) + 2b_3 \qquad (23)$$

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■ Example 3 (cont'd) – Cubic Splines

– The last condition comes from Eq. 18 as

i	x	$f(x)$
3	7	2.5
4	9	0.5

$$6a_1 x_i + 2b_1 = 0$$

$$6a_{n-1} x_n + 2b_{n-1} = 0$$

$$6a_1(3) + 2b_1 = 0$$

$$6a_3(9) + 2b_3 = 0 \qquad (24)$$

– Therefore, the 12 equations [4(4-1)] are as follows:

Interpolation Using Splines

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■ Example 3 (cont'd) – Cubic Splines

$$\begin{aligned}
27a_1 + 9b_1 + 3c_1 + d_1 &= 2.5 \\
91.125a_1 + 20.25b_1 + 4.5c_1 + d_1 &= 1 \\
91.125a_2 + 20.25b_2 + 4.5c_2 + d_2 &= 1 \\
343a_2 + 49b_2 + 7c_2 + d_2 &= 2.5 \\
343a_3 + 49b_3 + 7c_3 + d_3 &= 2.5 \\
729a_3 + 81b_3 + 9c_3 + d_3 &= 0.5 \\
60.75a_1 + 9b_1 + c_1 - 60.75a_2 - 9b_2 - c_2 &= 0 \\
147a_2 + 14b_2 + c_2 - 147a_3 - 14b_3 - c_3 &= 0 \\
27a_1 + 2b_1 - 27a_2 - 2b_2 &= 0 \\
42a_2 + 2b_2 - 42a_3 - 2b_3 &= 0 \\
18a_1 + 2b_1 &= 0 \\
54a_3 + 2b_3 &= 0
\end{aligned} \qquad (25)$$

Interpolation Using Splines

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- Example 3 (cont'd) – Cubic Splines

$$\begin{bmatrix} 27 & 9 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 91.125 & 20.25 & 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 91.125 & 20.25 & 4.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 343 & 49 & 7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 343 & 49 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 729 & 81 & 9 & 1 \\ 60.75 & 9 & 1 & 0 & -60.75 & -9 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 147 & 14 & 1 & 0 & -147 & -14 & -1 & 0 \\ 27 & 2 & 0 & 0 & -27 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 42 & 2 & 0 & 0 & -42 & -2 & 0 & 0 \\ 18 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 54 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Interpolation Using Splines

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- Example 3 (cont'd) – Cubic Splines
  - Therefore,

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0.172 \\ -1.547 \\ 3.253 \\ 2.020 \\ -0.177 \\ 3.163 \\ -17.941 \\ 33.813 \\ -0.0528 \\ 0.554 \\ 0.326 \\ -8.813 \end{bmatrix}$$

$$f_1(x) = 0.172x^3 - 1.547x^2 + 3.253x + 2.02 \qquad \text{for } 3 \le x \le 4.5$$

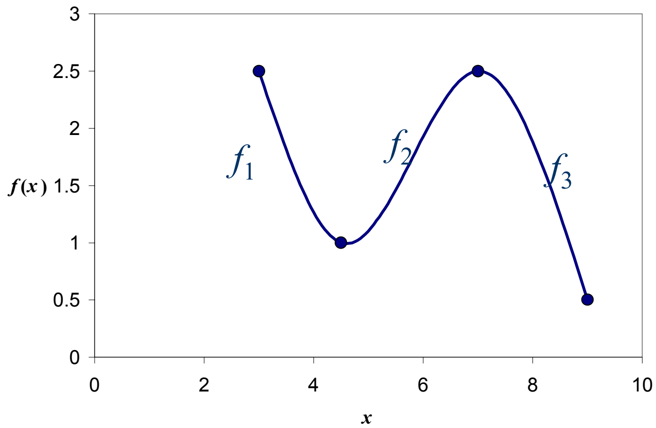
$$f_2(x) = -0.177x^3 + 3.163x^2 - 17.941x + 33.813 \qquad \text{for } 4.5 \le x \le 7$$

$$f_3(x) = -0.0528x^3 + 0.554x^2 + 0.326x - 8.813 \qquad \text{for } 7 \le x \le 9$$

$$\begin{aligned} f(5) &= -0.177(5)^3 + 3.163(5)^2 - 17.941(5) + 33.813 \\ &= 1.058 \end{aligned}$$

Interpolation Using Splines

- Example 3 (cont'd) – Cubic Splines



Interpolation Using Splines

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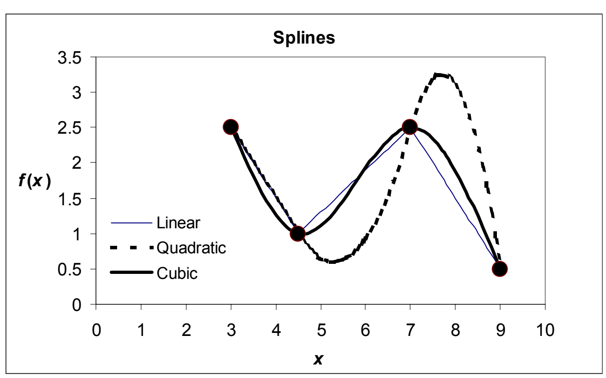
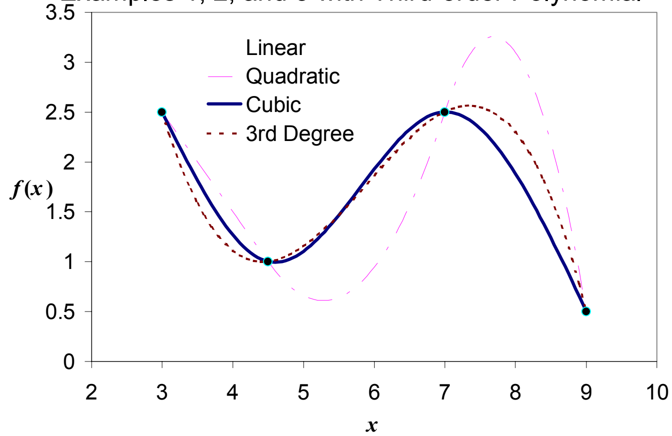
- Comparison among Linear, Quadratic, and Cubic Splines for Examples 1, 2, and 3



Interpolation Using Splines

- Comparison among Linear, Quadratic, and Cubic Splines for Examples 1, 2, and 3 with Third-order Polynomial



Multidimensional Interpolation

- In general, the function of interest can take the following form:

$$f(x_1, x_2, x_3, \ldots, x_n)$$

- In most engineering applications, two-dimensional interpolation is required.

Multidimensional Interpolation

- The previously introduced methods for one-dimensional case can be applied to multidimensional interpolation.
- However, with increased computational difficulties.
- To illustrate the concept, linear interpolation is introduced for two-dimensional interpolation.

Multidimensional Interpolation

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- Two-dimensional Interpolation
  - Example 4

    The following table gives the values of the gravitational acceleration, *g*, in feet per second squared as a function of altitude *L* (in degrees) and elevation *E* (in feet). Find the value of *g* when $L = 12.5^0$ and $E = 500$ feet.

Multidimensional Interpolation

- Two-dimensional Interpolation
  - Example 4

	$E = 0$ ft	$E = 1000$ ft	$E = 2000$ ft
$L = 0^0$	32.0877	32.0847	32.0816
$L = 5^0$	32.0890	32.0859	32.0829
$L = 10^0$	32.0928	32.0897	32.086
$L = 15^0$	32.0991	32.0960	32.0929
$L = 20^0$	32.1075	32.1044	32.1013

Multidimensional Interpolation

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- Two-dimensional Interpolation
  - Example 4 (cont'd)

$$f(x_1, x_2) = g(L, E)$$

- First, we interpolate for E at L = 10 to obtain $g(10^0$, 500 feet).

E	g
0	32.0928
500	g_1
1000	32.0897

$$\Rightarrow \frac{500-0}{1000-0} = \frac{g_1 - 32.0928}{32.0897 - 32.0928} \Rightarrow g_1 = 32.09125$$

Multidimensional Interpolation

- Two-dimensional Interpolation
  - Example 4 (cont'd)
    - Second, we interpolate for *E* at *L* = 15 to obtain *g*(15^0, 500 feet).

E	g
0	32.0991
500	g_1
1000	32.0960

$$\Rightarrow \frac{500-0}{1000-0} = \frac{g_2 - 32.0991}{32.0960 - 32.0991} \Rightarrow g_2 = 32.09755$$

Multidimensional Interpolation

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- Two-dimensional Interpolation
  - Example 4 (cont'd)
    - Using the values of g_1 and g_2, we interpolate for *L* at *E* = 500 to obtain *g*(12.5^0, 500 feet).

L	g
10	32.09125
12.5	g_3
15	32.09755

$$\Rightarrow \frac{12.5-10}{15-10} = \frac{g_2 - 32.09125}{32.09755 - 32.09125} \Rightarrow g_3 = 32.0944$$

    - Therefore, *g*(12.5^0, 500 feet) = g_3 = 32.0944 ft/s^2