CHAPTER 6d. NUMERICAL INTERPOLATION

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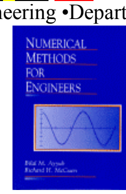

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering
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Finite Difference Interpolation

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- Example 4

  Repeat Example 2 using a finite difference table

x	1	2	3
$f(x)$	3	5	8

$\Delta x = 1$

Finite-difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	…	$\Delta^n f$
x	$f(x)$					
		$\Delta f(x)$				
$x+\Delta x$	$f(x+\Delta x)$		$\Delta^2 f(x)$			
		$\Delta f(x+\Delta x)$		$\Delta^3 f(x)$		
$x+2\Delta x$	$f(x+2\Delta x)$		$\Delta^2 f(x+\Delta x)$			
		$\Delta f(x+2\Delta x)$		$\Delta^3 f(x+\Delta x)$		
$x+3\Delta x$	$f(x+3\Delta x)$		$\Delta^2 f(x+2\Delta x)$			
		$\Delta f(x+3\Delta x)$		$\Delta^3 f(x+2\Delta x)$		
			$\Delta^2 f(x+3\Delta x)$			
:	:	:	:	:	:::	$\Delta^n f(x)$
$x+(n-2)\Delta x$	$f[x+(n-2)\Delta x]$		$\Delta^2 f[x+(n-3)\Delta x]$			
		$\Delta f[x+(n-2)\Delta x]$		$\Delta^3 f[x+(n-3)\Delta x]$		
$x+(n-1)\Delta x$	$f[x+(n-1)\Delta x]$		$\Delta^2 f[x+(n-2)\Delta x]$			
		$\Delta f[x+(n-1)\Delta x]$				
$x+n\Delta x$	$f(x+n\Delta x)$					

Finite Difference Interpolation

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- Example 4 (cont'd):

x	$f(x)$	Δf	$\Delta^2 f$
1	3		
		2	
2	5		1
		3	
3	8		

Finite Difference Interpolation

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- First and Second Finite Difference
  - The a quadratic polynomial of the form

$$f(x) = b_2 x^2 + b_1 x + b_0 \qquad (12)$$

  - The first and second finite difference are given as

$$\Delta f = 2b_2(\Delta x)x + \left(b_2(\Delta x)^2 + b_1 \Delta x\right) \qquad (17)$$

and

$$\Delta^2 f = 2b_2(\Delta x)^2 \qquad (18)$$

Finite Difference Interpolation

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- Example 4 (cont'd):

Equation 18 gives

$$\Delta^2 f = 2b_2(\Delta x)^2$$

$$1 = 2b_2(1)^2 \Rightarrow b_2 = 0.5$$

Equation 17 gives

$$\Delta f = 2b_2(\Delta x)x + \left(b_2(\Delta x)^2 + b_1 \Delta x\right)$$

$$2 = 2b_2(1)(1) + \left[b_2(1)^2 + (0.5)(1)\right]$$

$$\therefore b_2 = 0.5$$

Finite Difference Interpolation

- Example 4 (cont'd):

  Equation 12 gives

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$3 = 0.5(1)^2 + 0.5(1) + b_0$$

$$\therefore b_0 = 2$$

  Hence, the quadratic polynomial is

$$f(x) = 0.5x^2 + 0.5x + 2$$

Finite Difference Interpolation

- Example 4 (cont'd):

  Therefore, $f(2.7)$ can be estimated as

$$f(2.7) = 0.5(2.7)^2 + 0.5(2.7) + 2 = 6.995$$

Finite Difference Interpolation

- Newton's Method
  - Newton's method is a convenient algorithm to find an nth-order interpolation function with the use of a finite-difference table developed for a given set of data points.
  - Referring to general form of the finite-difference table, it can be shown that the first diagonal row are given by

$$f(x+\Delta x) = f(x) - \Delta f(x)$$

Finite-difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	…	$\Delta^n f$
x	$f(x)$					
		$\Delta f(x)$				
$x+\Delta x$	$f(x+\Delta x)$		$\Delta^2 f(x)$			
		$\Delta f(x+\Delta x)$		$\Delta^3 f(x)$		
$x+2\Delta x$	$f(x+2\Delta x)$		$\Delta^2 f(x+\Delta x)$			
		$\Delta f(x+2\Delta x)$		$\Delta^3 f(x+\Delta x)$		
$x+3\Delta x$	$f(x+3\Delta x)$		$\Delta^2 f(x+2\Delta x)$			
		$\Delta f(x+3\Delta x)$		$\Delta^3 f(x+2\Delta x)$		
			$\Delta^2 f(x+3\Delta x)$			
:	:	:	:	:	:::	$\Delta^n f(x)$
$x+(n-2)\Delta x$	$f[x+(n-2)\Delta x]$		$\Delta^2 f[x+(n-3)\Delta x]$			
		$\Delta f[x+(n-2)\Delta x]$		$\Delta^3 f[x+(n-3)\Delta x]$		
$x+(n-1)\Delta x$	$f[x+(n-1)\Delta x]$		$\Delta^2 f[x+(n-2)\Delta x]$			
		$\Delta f[x+(n-1)\Delta x]$				
$x+n\Delta x$	$f(x+n\Delta x)$					

Finite Difference Interpolation

- Newton's Method

From the finite-difference table, in the first diagonal, we notice the following:

$$\begin{aligned}\Delta f(x) &= f(x+\Delta x) - f(x)\\ \Delta^2 f(x) &= \Delta f(x+\Delta x) - \Delta f(x)\\ \Delta^3 f(x) &= \Delta^2 f(x+\Delta x) - \Delta^2 f(x)\\ &\vdots\\ \Delta^n f(x) &= \Delta^{n-1} f(x+\Delta x) - \Delta^{n-1} f(x)\end{aligned} \qquad (21)$$

Finite Difference Interpolation

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- Newton's Method

Equation 21 can be rearrange to give

$$\begin{aligned}f(x+\Delta x) &= f(x) + \Delta f(x)\\ \Delta f(x+\Delta x) &= \Delta f(x) + \Delta^2 f(x)\\ \Delta^2 f(x+\Delta x) &= \Delta^2 f(x) + \Delta^3 f(x)\\ &\vdots\\ \Delta^{n-1} f(x+\Delta x) &= \Delta^{n-1} f(x) + \Delta^n f(x)\end{aligned} \qquad (22)$$

Finite Difference Interpolation

■ Newton's Method

The same procedure could be used for any diagonal row. For example, the equations for the next diagonal row are

$$
\begin{aligned}
f(x+2\Delta x) &= f(x) + 2\Delta f(x) + \Delta^2 f(x) \\
\Delta f(x+2\Delta x) &= \Delta f(x) + 2\Delta^2 f(x) + \Delta^3 f(x) \\
\Delta^2 f(x+2\Delta x) &= \Delta^2 f(x) + 2\Delta^3 f(x) + \Delta^4 f(x) \\
&\vdots \\
\Delta^{n-3} f(x+2\Delta x) &= \Delta^{n-3} f(x) + 2\Delta^{n-2} f(x) + \Delta^{n-1} f(x)
\end{aligned}
\qquad (23)
$$

Finite Difference Interpolation

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■ Newton's Method

All these expansions have the form of binomial expansion; thus they can be rewritten as

$$\Delta f(x+m\Delta x) = \sum_{i=0}^{m} b_i \Delta^i f(x) \qquad (24)$$

$$\Delta f(x+m\Delta x) = f(x) + m\Delta f(x) + \frac{m(m-1)}{2}\Delta^2 f(x) + \cdots + \Delta^m f(x)$$

Finite Difference Interpolation

- Newton's Method

  In general

$$\Delta^r f(x+m\Delta x) = \sum_{i=0}^{m} b_i \Delta^{r+`} f(x) \qquad (25)$$

$$\Delta^r f(x+m\Delta x) = \Delta^r f(x) + m\Delta^{r+1} f(x) + \cdots + \Delta^{r+m} f(x)$$

Finite Difference Interpolation

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- Newton's Interpolation Formula

  Previous equation fives the following formula:

$$f(x) = f(x_0) + n\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{n(n-1)\cdots(n-m+1)}{m!}\Delta^m f(x_0) \qquad (26)$$

in which

$$n = \frac{x - x_0}{\Delta x} \qquad (27)$$

Finite Difference Interpolation

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- Example 5

Repeat example 4 using Newton's formula.

x	1	2	3
$f(x)$	3	5	8

$\Delta x = 1$

Finite Difference Interpolation

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- Example 5 (cont'd):

The following finite-difference table can be constructed:

x	$f(x)$	Δf	$\Delta^2 f$
1	3		
		2	
2	5		1
		3	
3	8		

Finite Difference Interpolation

■ Example 5 (cont'd):

The Newton's formula in this case will be written as

$$f(x) = f(x_0) + n\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0) \qquad (28)$$

where the following values can be obtained from the finite-difference table:

$$x_0 = 1,\ f(1) = 3,\ \Delta f(1) = 2, \text{ and } \Delta^2 f(1) = 1$$

Finite Difference Interpolation

■ Example 5 (cont'd):

$x_0 = 1$

$f(x_0) = f(1) = 3$

$\Delta f(x_0) = \Delta f(1) = 2$

$\Delta^2 f(x_0) = \Delta^2 f(1) = 1$

i	x	$f(x)$	Δf	$\Delta^2 f$
0	1	3		
			2	
1	2	5		1
			3	
2	3	8		

Finite Difference Interpolation

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■ Example 5 (cont'd):

Therefore, Equation 28 can be expressed as

$$f(x) = 3 + n(2) + \frac{n(n-1)}{2}(1) \qquad (29)$$

$x_0 = 1$
$f(x_0) = f(1) = 3$
$\Delta f(x_0) = \Delta f(1) = 2$
$\Delta^2 f(x_0) = \Delta^2 f(1) = 1$

$$f(x) = f(x_0) + n\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0)$$

Finite Difference Interpolation

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■ Example 5 (cont'd):

Equation 29 gives the quadratic interpolation function for this example:

$$f(x) = 3 + 2n + \frac{n(n-1)}{2}$$

To estimate $f(2.7)$, we need to find n from Eq. 27 as follows

$$n = \frac{x - x_0}{\Delta x} = \frac{2.7 - 1}{1} = 1.7$$

$$f(x) = 3 + 2(1.7) + \frac{1.7(1.7-1)}{2} = 6.995$$

Finite Difference Interpolation

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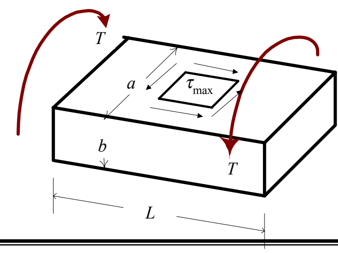
■ Example 6

Use the Newton's interpolation formula to find the angle of twist and the largest torque which may be applied to the noncircular brass bar as shown. Assume $\tau = 40\times 10^6$ Pa, $a = 0.064$ m, $b = 0.025$ m, $L = 2$ m, and $G = 77\times10^9$ Pa.

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$\phi = \frac{TL}{c_2 a b^3 G}$$



Finite Difference Interpolation

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■ Example 6 (cont'd)

The following table gives the constant c_1 and c_2 values

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$\phi = \frac{TL}{c_2 a b^3 G}$$

a/b	c_1	c_2
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
∞	0.333	0.333

Finite Difference Interpolation

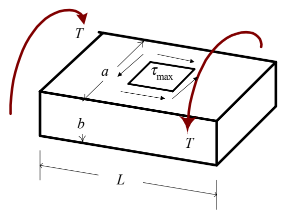
- Example 6 (cont'd)

  We need to find two interpolation functions for c_1 and c_2 using Newton's formula.

  This will require two finite-difference tables for c_1 and c_2 as follows:

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$\phi = \frac{TL}{c_2 a b^3 G}$$



Finite Difference Interpolation

- Example 6 (cont'd)

  Finite-difference Table for c_1

a/b	c_1	Δc_1	$\Delta^2 c_1$	$\Delta^3 c_1$
1.5	0.231			
		0.015		
2.0	0.246		-0.003	
		0.012		0
2.5	0.258		-0.003	
		0.009		
3.0	0.267			

Finite Difference Interpolation

- Example 6 (cont'd)

Finite-difference Table for c_2

a/b	c_2	Δc_2	$\Delta^2 c_2$	$\Delta^3 c_2$
1.5	0.196			
		0.033		
2.0	0.229		-0.013	
		0.020		0.007
2.5	0.249		-0.006	
		0.014		
3.0	0.263			

Finite Difference Interpolation

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- Example 6 (cont'd)

For c_1 and c_2, the Newton's interpolation formula of Eq. 26 gives

$$f(x) = f(x_0) + n\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0) + \frac{n(n-1)(n-2)}{3!}\Delta^3 f(x_0) \quad (30)$$

Finite Difference Interpolation

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■ Example 6 (cont'd)

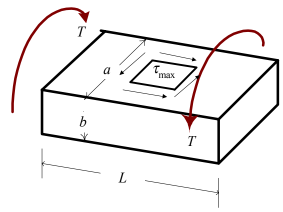
Finding the Largest Torque:

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56 = x$$

$$\left(\frac{a}{b}\right)_0 = x_0 = 1.5$$

$$\Delta\left(\frac{a}{b}\right) = \Delta x = 2 - 1.5 = 0.5$$

$$n \frac{x - x_0}{\Delta x} = \frac{2.56 - 1.5}{0.5} = 2.120$$



Finite Difference Interpolation

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■ Example 6 (cont'd)

Finding the Largest Torque:

From the finite-difference table for c_1

$$f\left(\left[\frac{a}{b}\right]_0\right) = f(x_0) = c_1(1.5) = 0.231$$

$$\Delta f\left(\left[\frac{a}{b}\right]_0\right) = \Delta f(x_0) = \Delta c_1(1.5) = 0.015$$

$$\Delta^2 f\left(\left[\frac{a}{b}\right]_0\right) = \Delta^2 f(x_0) = \Delta^2 c_1(1.5) = -0.003$$

$$\Delta^3 f\left(\left[\frac{a}{b}\right]_0\right) = \Delta^3 f(x_0) = \Delta^3 c_1(1.5) = 0$$

Finite Difference Interpolation

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- Example 6 (cont'd)

  Substituting above values into Newton's interpolation formula of Eq. 30, we have

$$f(x) = 0.231 + n(0.015) + \frac{n(n-1)}{2}(-0.003) + \frac{n(n-1)(n-2)}{6}(0)$$

or

$$c_1 = f(x) = 0.231 + 0.015n - 0.003\frac{n(n-1)}{2}$$

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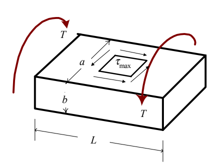
- Example 6 (cont'd)



  Therefore

For $a/b = 2.56$, $n = 2.12$

$$c_1(2.12) = 0.231 + 0.015(2.12) - 0.003\frac{2.12(2.12-1)}{2} = 0.25924$$

  and

$$\tau_{\max} = \frac{T}{c_1 ab^2}$$

$$40\times10^6 = \frac{T}{0.25924(0.064)(0.025)^2} \Rightarrow T = \underline{414.8 \text{ N.m}}$$

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- Example 6 (cont'd)

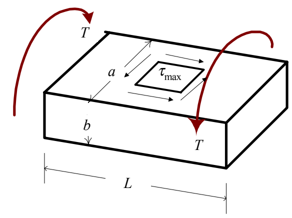
Finding the Angle of Twist:

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56 = x$$

$$\left(\frac{a}{b}\right)_0 = x_0 = 1.5$$

$$\Delta\left(\frac{a}{b}\right) = \Delta x = 2 - 1.5 = 0.5$$

$$n\frac{x - x_0}{\Delta x} = \frac{2.56 - 1.5}{0.5} = 2.120$$



Finite Difference Interpolation

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- Example 6 (cont'd)

Finding the Angle of Twist:

From the finite-difference table for c_2

$$f\left(\left[\frac{a}{b}\right]_0\right) = f(x_0) = c_2(1.5) = 0.196$$

$$\Delta f\left(\left[\frac{a}{b}\right]_0\right) = \Delta f(x_0) = \Delta c_2(1.5) = 0.033$$

$$\Delta^2 f\left(\left[\frac{a}{b}\right]_0\right) = \Delta^2 f(x_0) = \Delta^2 c_2(1.5) = -0.013$$

$$\Delta^3 f\left(\left[\frac{a}{b}\right]_0\right) = \Delta^3 f(x_0) = \Delta^3 c_2(1.5) = 0.007$$

Finite Difference Interpolation

■ Example 6 (cont'd)

Finite-difference Table for c_2

a/b	c_2	Δc_2	$\Delta^2 c_2$	$\Delta^3 c_2$
1.5	0.196			
		0.033		
2.0	0.229		-0.013	
		0.020		0.007
2.5	0.249		-0.006	
		0.014		
3.0	0.263			

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■ Example 6 (cont'd)

Substituting above values into Newton's interpolation formula of Eq. 30, we have

$$f(x) = 0.196 + n(0.033) + \frac{n(n-1)}{2}(-0.013) + \frac{n(n-1)(n-2)}{6}(0.007)$$

or

$$c_2 = f(x) = 0.196 + 0.033n - 0.013\frac{n(n-1)}{2} + 0.007\frac{n(n-1)(n-2)}{6}$$

Finite Difference Interpolation

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■ Example 6 (cont'd)

Therefore,

For $a/b = 2.56, n = 2.12$

$$c_2(2.12) = 0.196 + 0.033(2.12) - 0.013\frac{2.12(2.12-1)}{2} + 0.007\frac{2.12(2.12-1)(2.12-2)}{6}$$

$$= 0.25086$$

Finite Difference Interpolation

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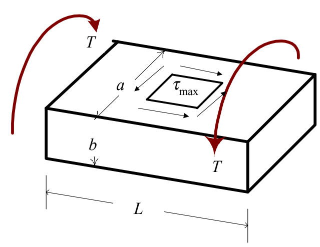
■ Example 6 (cont'd)

Therefore,



$$\phi = \frac{TL}{c_2 ab^3 G}$$

$$\phi = \frac{414.8(2)}{0.25086(0.064)(0.025)^3(77\times10^9)}$$

$$= \underline{0.04295} = \text{Angle of Twist}$$

Lagrange Polynomials

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- Lagrange interpolation polynomials are useful when the given data contains unequal intervals for the independent variables x.
- In fact, this is the case for many engineering problems.
- The data collection cannot be controlled.

Lagrange Polynomials

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- The data are collected for one variables $f(x)$, which is the independent variables, as a function of second variable x.
- In the Newton's interpolation formula, the independent variable x was assumed to be measured at a constant interval, Δx.
- Lagrange method can handled problem with a varying Δx.

Lagrange Polynomials

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■ Definition

The *Lagrange interpolation polynomial* is simply a reformulation of Newton's polynomial that avoids the computation using finite difference, and that can handled problems with varying interval.

Lagrange Polynomials

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■ Definition of Terms

- The data sample is assumed to consist of n pairs of values measured on x and $f(x)$, with x_i being the ith measured value of the independent variable.
- The method provides an estimate of the value of $f(x)$ at a specified value of x, which is denoted x_0.

Lagrange Polynomials

- Definition of Terms

  $f(x_0)$ = estimated value

  $f(x_i)$ = measured value of dependent variable

  i = 1, 2, 3, …, n

  The method involves a weighting function, with the weight given to the ith value of f for x_0 denoted as $w_i(x_0)$.

Lagrange Polynomials

- Lagrange Polynomial

  The Lagrange interpolating polynomial for estimating the value of $f(x_0)$ is given by

$$f(x_0) = \sum_{i=1}^{n} w_i(x_0) f(x_i) \tag{31}$$

Lagrange Polynomials

- Lagrange Polynomial

Where

$$w_i(x_i) = \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n-1}(x_0 - x_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{n-1}(x_i - x_j)} \quad (32)$$

$f(x_0)$ = estimated value
$f(x_i)$ = measured value of dependent variable
i = 1, 2, 3, …, n
$w_i(x_0)$ = weighting function
Π = the product of

Lagrange Polynomials

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- Lagrange Polynomial

Combining Equations 31 and 32, the result is

$$f(x_0) = w_1(x_0)f(x_1) + w_2(x_0)f(x_2) + w_3(x_0)f(x_3) + \ldots + w_n(x_0)f(x_n) \quad (33)$$