

# CHAPTER 6c. NUMERICAL INTERPOLATION



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## Method of Undetermined Coefficients



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### ■ Example 6

- Develop a fourth-order interpolation polynomial for the following set of data, for which we know their original function, that is,  $f(x) = x^3$ .

|        |   |   |   |    |    |
|--------|---|---|---|----|----|
| $x$    | 0 | 1 | 2 | 3  | 4  |
| $f(x)$ | 0 | 1 | 8 | 27 | 64 |



## Finite Difference Interpolation

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- If the values of the independent variables are equally spaced, a finite difference scheme can be used to develop an interpolation polynomial.
  - If  $\Delta x$  is the increment on which the values of the independent variable are recorded, then the first finite difference of the dependent variable  $y = f(x)$  is

$$\Delta f = f(x + \Delta x) - f(x) \quad (1)$$



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- The second finite difference can be expressed as

$$\Delta^2 f = \Delta[\Delta f] = [f(x + 2\Delta x) - f(x + \Delta x)] - [f(x + \Delta x) - f(x)] \quad (2)$$

- In general, the  $n$  finite difference is

$$\Delta^n f = \Delta[\Delta^{n-1} f] \quad (3)$$



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## ■ Rule

- The  $i$ th finite difference, where  $i$  is less than or equal to  $n$ , of an  $n$ th-degree polynomial is a polynomial of degree  $(n - 1)$ .
- This rule implies that the  $n$ th difference would be a constant.
- Assume the following  $n$ th-order polynomial:

$$f(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1} + b_nx^n \quad (4)$$

$b_n$  does not equal zero.



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- For  $(x + \Delta x)$ , the polynomial given by Eq. 4 becomes

$$f(x + \Delta x) = b_0 + b_1(x + \Delta x) + b_2(x + \Delta x)^2 + b_3(x + \Delta x)^3 + \dots + b_n(x + \Delta x)^n \quad (5)$$

- This polynomial can be rearrange into

$$f(x + \Delta x) = b_nx^n + (b_{n-1} + n\Delta xb_n)x^{n-1} + \dots \quad (6)$$



# Finite Difference Interpolation

- The first finite difference can be constructed by subtracting Eq. 4 from Eq.6 as follows:

$$f(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1} + b_nx^n$$

$$f(x + \Delta x) = b_nx^n + (b_{n-1} + n\Delta xb_n)^{n-1} + \dots$$

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$$\Delta f = f(x + \Delta x) - f(x)$$

Or

$$\Delta f = (b_{n-1} + n\Delta xb_n)x^{n-1} + \dots$$

(7)



# Finite Difference Interpolation

- For  $(x + 2\Delta x)$ , Eq. 5 becomes

$$f(x + 2\Delta x) = b_n(x + 2\Delta x)^n + b_{n-1}(x + 2\Delta x)^{n-1} + \dots \quad (8)$$

- Equations 8, 6, and 4 can be substituted into Equation 2 to compute a second finite difference, with the following result:

$$\Delta^2 f = n(n-1)(\Delta x)^2 b_n x^{n-2} + \dots \quad (9)$$



# Finite Difference Interpolation

- Equation 9 is a polynomial of order  $(n-2)$ .
- By induction, the  $i$ th finite difference is given by

$$\Delta^i f = n(n-1)(n-2)\cdots(n-i+1)(\Delta x)^i b_n x^{n-i} + \cdots \quad (10)$$

- Equation 10 is a polynomial of order  $(n-1)$ .
- The  $n$ th finite difference equals the following constant:

$$\Delta^n f = n(n-1)(n-2)\cdots(1)(\Delta x)^n b_n = n!(\Delta x)^n b_n \quad (11)$$



# Finite Difference Interpolation

## ■ Example 1

- Suppose that  $n = 2$ , then Eq. 4 becomes

$$f(x) = b_0 + b_1x + b_2x^2$$

Or

$$f(x) = b_2x^2 + b_1x + b_0 \quad (12)$$



# Finite Difference Interpolation

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## ■ Example 1 (cont'd):

– For  $(x + \Delta x)$ ,  $f(x + \Delta x)$  can be defined as follows:

$$\begin{aligned}
 f(x + \Delta x) &= b_2(x + \Delta x)^2 + b_1(x + \Delta x) + b_0 \\
 &= b_2(x^2 + 2x\Delta x + (\Delta x)^2) + b_1(x + \Delta x) + b_0 \\
 &= b_2x^2 + b_2 2x\Delta x + b_2(\Delta x)^2 + b_1x + b_1\Delta x + b_0 \\
 &= b_2x^2 + (2\Delta x b_2 + b_1)x + (b_2(\Delta x)^2 + b_1\Delta x + b_0)
 \end{aligned}$$

$$f(x + \Delta x) = b_2x^2 + (2\Delta x b_2 + b_1)x + (b_2(\Delta x)^2 + b_1\Delta x + b_0) \quad (13)$$



# Finite Difference Interpolation

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## ■ Example 1 (cont'd):

– For  $(x + 2\Delta x)$ ,  $f(x + 2\Delta x)$  can be defined as follows:

$$\begin{aligned}
 f(x + 2\Delta x) &= b_2(x + 2\Delta x)^2 + b_1(x + 2\Delta x) + b_0 \\
 &= b_2(x^2 + 4x\Delta x + (2\Delta x)^2) + b_1(x + 2\Delta x) + b_0 \\
 &= b_2x^2 + b_2 4x\Delta x + b_2(2\Delta x)^2 + b_1x + 2b_1\Delta x + b_0 \\
 &= b_2x^2 + (4\Delta x b_2 + b_1)x + (b_2(2\Delta x)^2 + 2b_1\Delta x + b_0)
 \end{aligned}$$

$$f(x + 2\Delta x) = b_2x^2 + (4\Delta x b_2 + b_1)x + (b_2(2\Delta x)^2 + 2b_1\Delta x + b_0) \quad (14)$$



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## ■ Example 1 (cont'd):

– Recall Eq. 1, 12, and 13

$$\Delta f = f(x + \Delta x) - f(x)$$

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$f(x + \Delta x) = b_2 x^2 + (2\Delta x b_2 + b_1)x + (b_2 (\Delta x)^2 + b_1 \Delta x + b_0)$$

The first difference is computed as follows:

$$\Delta f = 2b_2 (\Delta x)x + (b_2 (\Delta x)^2 + b_1 \Delta x) \quad (15)$$



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## ■ Example 1 (cont'd):

– Similarly, recall Eq. 2, 12, 13, and 14

$$\Delta^2 f = \Delta[\Delta f] = [f(x + 2\Delta x) - f(x + \Delta x)] - [f(x + \Delta x) - f(x)]$$

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$f(x + 2\Delta x) = b_2 x^2 + (4\Delta x b_2 + b_1)x + (b_2 (2\Delta x)^2 + 2b_1 \Delta x + b_0)$$

$$f(x + \Delta x) = b_2 x^2 + (2\Delta x b_2 + b_1)x + (b_2 (\Delta x)^2 + b_1 \Delta x + b_0)$$

The second finite difference is computed as follows:

$$\Delta^2 f = 2b_2 (\Delta x)^2 \quad (16)$$



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## ■ First and Second Finite Difference

- The a quadratic polynomial of the form

$$f(x) = b_2x^2 + b_1x + b_0$$

- The first and second finite difference are given as

$$\Delta f = 2b_2(\Delta x)x + (b_2(\Delta x)^2 + b_1\Delta x) \quad (17)$$

and

$$\Delta^2 f = 2b_2(\Delta x)^2 \quad (18)$$



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## ■ Example 2

- Develop an interpolation polynomial for the following data using the finite difference approach. Estimate the  $f(x)$  for  $x = 2.7$

|        |   |   |   |
|--------|---|---|---|
| $x$    | 1 | 2 | 3 |
| $f(x)$ | 3 | 5 | 8 |

$$\Delta x = 2 - 1 = 3 - 2 = 1$$





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## ■ Example 2 (cont'd):

|        |   |   |   |
|--------|---|---|---|
| $x$    | 1 | 2 | 3 |
| $f(x)$ | 3 | 5 | 8 |

$$\Delta x = 1$$

$$f(x) = 3$$

$$f(x + \Delta x) = 5$$



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## ■ Example 2 (cont'd):

- The first finite difference can be obtained using Eq. 1 as

$$\Delta f = f(x + \Delta x) - f(x) = 5 - 3 = 2$$

- This value can be equated to Eq. 17 as follows:

$$\Delta f = 2b_2(\Delta x)x + (b_2(\Delta x)^2 + b_1\Delta x)$$

$$2 = 2b_2(1)(1) + (b_2(1)^2 + b_1(1))$$

$$2 = 3b_2 + b_1 \quad (19)$$



## Finite Difference Interpolation

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### ■ Example 2 (cont'd):

- The second finite difference can be obtained using Eq. 2 as

$$\begin{aligned}\Delta^2 f &= [f(x + 2\Delta x) - f(x + \Delta x)] - [f(x + \Delta x) - f(x)] \\ &= [8 - 5] - [5 - 3] = 1\end{aligned}$$

- This value can be equated to Eq. 18 as follows:

$$\begin{aligned}\Delta^2 f &= 2b_2(\Delta x)^2 \\ 1 &= 2b_2(1)^2 = 2b_2\end{aligned}\quad (20)$$



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### ■ Example 2 (cont'd):

- Solving Eq. 20, gives  $b_2 = 0.5$
- Substituting this result into Eq. 19, gives

$$2 = 3b_2 + b_1 \Rightarrow b_1 = 2 - 3b_2 = 2 - 3(0.5)$$

Or

$$b_1 = 0.5$$



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### ■ Example 2 (cont'd):

- Using these values of  $b_1$  and  $b_2$  and any one of the three points, a value of  $b_0$  can be calculated from Eq. 12 as follows:

$$f(x) = b_2x^2 + b_1x + b_0$$

$$\begin{aligned} b_0 &= f(x) = 8 - 0.5(3)^2 - 0.5(3) \\ &= 2 \end{aligned}$$



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### ■ Example 2 (cont'd):

- Thus, the interpolation polynomial is

$$f(x) = b_2x^2 + b_1x + b_0$$

$$f(x) = 0.5x^2 + 0.5x + 2$$

- $f(2.7)$  can be estimated:

$$f(2.7) = 0.5(2.7)^2 + 0.5(2.7) + 2 = 6.995$$

# Finite Difference Interpolation



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## ■ Finite-difference Tables

- We see in the preceding example that the finite difference approach can be used to construct an interpolation polynomial.
- However, the procedure can be tedious and requires a lot of computations.
- Therefore, we need some sort of scheme to organize the procedure using finite-difference tables.

# Finite Difference Interpolation



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## ■ Finite-difference Table

- The finite-differences can be organized into a table as shown in the next viewgraph.
- The finite-difference table can be used to derive an interpolation polynomial.
- As we will see, finite-difference tables can be used with Newton formula to derive interpolation polynomials.

### Finite-difference Table

| $x$                 | $f(x)$                 | $\Delta f$                    | $\Delta^2 f$                    | $\Delta^3 f$                    | ... | $\Delta^n f$    |
|---------------------|------------------------|-------------------------------|---------------------------------|---------------------------------|-----|-----------------|
| $x$                 | $f(x)$                 |                               |                                 |                                 |     |                 |
|                     |                        | $\Delta f(x)$                 |                                 |                                 |     |                 |
| $x + \Delta x$      | $f(x + \Delta x)$      |                               | $\Delta^2 f(x)$                 |                                 |     |                 |
|                     |                        | $\Delta f(x + \Delta x)$      |                                 | $\Delta^3 f(x)$                 |     |                 |
| $x + 2\Delta x$     | $f(x + 2\Delta x)$     |                               | $\Delta^2 f(x + \Delta x)$      |                                 |     |                 |
|                     |                        | $\Delta f(x + 2\Delta x)$     |                                 | $\Delta^3 f(x + \Delta x)$      |     |                 |
| $x + 3\Delta x$     | $f(x + 3\Delta x)$     |                               | $\Delta^2 f(x + 2\Delta x)$     |                                 |     |                 |
|                     |                        | $\Delta f(x + 3\Delta x)$     |                                 | $\Delta^3 f(x + 2\Delta x)$     |     |                 |
|                     |                        |                               | $\Delta^2 f(x + 3\Delta x)$     |                                 |     |                 |
| :                   | :                      | :                             | :                               | :                               | ... | $\Delta^n f(x)$ |
| $x + (n-2)\Delta x$ | $f[x + (n-2)\Delta x]$ |                               | $\Delta^2 f[x + (n-3)\Delta x]$ |                                 |     |                 |
|                     |                        | $\Delta f[x + (n-2)\Delta x]$ |                                 | $\Delta^3 f[x + (n-3)\Delta x]$ |     |                 |
| $x + (n-1)\Delta x$ | $f[x + (n-1)\Delta x]$ |                               | $\Delta^2 f[x + (n-2)\Delta x]$ |                                 |     |                 |
|                     |                        | $\Delta f[x + (n-1)\Delta x]$ |                                 |                                 |     |                 |
| $x + n\Delta x$     | $f(x + n\Delta x)$     |                               |                                 |                                 |     |                 |

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### ■ Example 3

Construct a finite-difference table for the following set of data:

|        |      |      |      |      |      |
|--------|------|------|------|------|------|
| $x$    | 10   | 11   | 12   | 13   | 14   |
| $f(x)$ | 1000 | 1331 | 1728 | 2197 | 2744 |

# Finite Difference Interpolation



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## ■ Example 3 (cont'd):

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ |
|-----|--------|------------|--------------|--------------|--------------|
| 10  | 1000   |            |              |              |              |
|     |        | 331        |              |              |              |
| 11  | 1331   |            | 66           |              |              |
|     |        | 397        |              | 6            |              |
| 12  | 1728   |            | 72           |              | 0            |
|     |        | 469        |              | 6            |              |
| 13  | 2197   |            | 78           |              |              |
|     |        | 547        |              |              |              |
| 14  | 2744   |            |              |              |              |