# CHAPTER 6c. NUMERICAL INTERPOLATION

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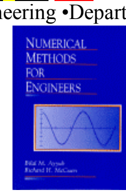


by

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# Method of Undetermined Coefficients

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- Example 6
  - Develop a fourth-order interpolation polynomial for the following set of data, for which we know their original function, that is, $f(x) = x^3$.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | 8 | 27 | 64 |


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# Finite Difference Interpolation



- If the values of the independent variables are equally spaced, a finite difference scheme can be used to develop an interpolation polynomial.
  - If $\Delta x$ is the increment on which the values of the independent variable are recorded, then the first finite difference of the dependent variable $y = f(x)$ is

$$\Delta f = f(x + \Delta x) - f(x) \qquad (1)$$



# Finite Difference Interpolation



- The second finite difference can be expressed as

$$\Delta^2 f = \Delta[\Delta f] = [f(x + 2\Delta x) - f(x + \Delta x)] - [f(x + \Delta x) - f(x)] \qquad (2)$$

- In general, the $n$ finite difference is

$$\Delta^n f = \Delta\left[\Delta^{n-1} f\right] \qquad (3)$$

# Finite Difference Interpolation

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- Rule
  - The *i*th finite difference, where *i* is less than or equal to *n*, of an *n*th-degree polynomial is a polynomial of degree (*n* – 1).
  - This rule implies that the *n*th difference would be a constant.
  - Assume the following *n*th-order polynomial:

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} + b_n x^n \qquad (4)$$

$b_n$ does not equal zero.



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- For $(x + \Delta x)$, the polynomial given by Eq. 4 becomes

$$f(x+\Delta x) = b_0 + b_1(x+\Delta x) + b_2(x+\Delta x)^2 + b_3(x+\Delta x)^3 + \cdots + b_n(x+\Delta x)^n \qquad (5)$$

- This polynomial can be rearrange into

$$f(x+\Delta x) = b_n x^n + (b_{n-1} + n\Delta x b_n)^{n-1} + \cdots \qquad (6)$$

# Finite Difference Interpolation



– The first finite difference can be constructed by subtracting Eq. 4 from Eq.6 as follows:

$$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1} + b_n x^n$$

$$f(x + \Delta x) = b_n x^n + (b_{n-1} + n\Delta x b_n)^{n-1} + \cdots$$

$$\Delta f = f(x + \Delta x) - f(x)$$

Or

$$\Delta f = (b_{n-1} + n\Delta x b_n) x^{n-1} + \cdots \quad (7)$$



# Finite Difference Interpolation



– For $(x + 2\Delta x)$, Eq. 5 becomes

$$f(x + 2\Delta x) = b_n (x + 2\Delta x)^n + b_{n-1} (x + 2\Delta x)^{n-1} + \cdots \quad (8)$$

– Equations 8, 6, and 4 can be substituted into Equation 2 to compute a second finite difference, with the following result:

$$\Delta^2 f = n(n-1)(\Delta x)^2 b_n x^{n-2} + \cdots \quad (9)$$

# Finite Difference Interpolation

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– Equation 9 is a polynomial of order ($n$-2).
– By induction, the $i$th finite difference is given by

$$\Delta^i f = n(n-1)(n-2)\cdots(n-i+1)(\Delta x)^i b_n x^{n-i} + \cdots \qquad (10)$$

– Equation 10 is a polynomial of order ($n$ –1).
– The $n$th finite difference equals the following constant:

$$\Delta^n f = n(n-1)(n-2)\cdots(1)(\Delta x)^n b_n = n!(\Delta x)^n b_n \qquad (11)$$



# Finite Difference Interpolation

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■ Example 1

– Suppose that $n$ = 2, then Eq. 4 becomes

$$f(x) = b_0 + b_1 x + b_2 x^2$$

Or

$$f(x) = b_2 x^2 + b_1 x + b_0 \qquad (12)$$

# Finite Difference Interpolation

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- Example 1 (cont'd):
  - For $(x + \Delta x)$, $f(x + \Delta x)$ can be defined as follows:

$$
\begin{aligned}
f(x+\Delta x) &= b_2(x+\Delta x)^2 + b_1(x+\Delta x) + b_0 \\
&= b_2\left(x^2 + 2x\Delta x + (\Delta x)^2\right) + b_1(x+\Delta x) + b_0 \\
&= b_2x^2 + b_2 2x\Delta x + b_2(\Delta x)^2 + b_1x + b_1\Delta x + b_0 \\
&= b_2x^2 + (2\Delta x b_2 + b_1)x + \left(b_2(\Delta x)^2 + b_1\Delta x + b_0\right)
\end{aligned}
$$

$$\boxed{f(x+\Delta x) = b_2x^2 + (2\Delta x b_2 + b_1)x + \left(b_2(\Delta x)^2 + b_1\Delta x + b_0\right)} \quad (13)$$



# Finite Difference Interpolation

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- Example 1 (cont'd):
  - For $(x + 2\Delta x)$, $f(x + 2\Delta x)$ can be defined as follows:

$$
\begin{aligned}
f(x+2\Delta x) &= b_2(x+2\Delta x)^2 + b_1(x+2\Delta x) + b_0 \\
&= b_2\left(x^2 + 4x\Delta x + (2\Delta x)^2\right) + b_1(x+2\Delta x) + b_0 \\
&= b_2x^2 + b_2 4x\Delta x + b_2(2\Delta x)^2 + b_1x + 2b_1\Delta x + b_0 \\
&= b_2x^2 + (4\Delta x b_2 + b_1)x + \left(b_2(2\Delta x)^2 + 2b_1\Delta x + b_0\right)
\end{aligned}
$$

$$\boxed{f(x+2\Delta x) = b_2x^2 + (4\Delta x b_2 + b_1)x + \left(b_2(2\Delta x)^2 + 2b_1\Delta x + b_0\right)} \quad (14)$$

# Finite Difference Interpolation



- Example 1 (cont'd):
  - Recall Eq. 1, 12, and 13

$$\Delta f = f(x+\Delta x) - f(x)$$

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$f(x+\Delta x) = b_2 x^2 + (2\Delta x b_2 + b_1)x + \left(b_2(\Delta x)^2 + b_1 \Delta x + b_0\right)$$

The first difference is computed as follows:

$$\Delta f = 2b_2(\Delta x)x + \left(b_2(\Delta x)^2 + b_1 \Delta x\right) \qquad (15)$$



# Finite Difference Interpolation



- Example 1 (cont'd):
  - Similarly, recall Eq. 2,12,13, and 14

$$\Delta^2 f = \Delta[\Delta f] = [f(x+2\Delta x) - f(x+\Delta x)] - [f(x+\Delta x) - f(x)]$$

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$f(x+2\Delta x) = b_2 x^2 + (4\Delta x b_2 + b_1)x + \left(b_2(2\Delta x)^2 + 2b_1 \Delta x + b_0\right)$$

$$f(x+\Delta x) = b_2 x^2 + (2\Delta x b_2 + b_1)x + \left(b_2(\Delta x)^2 + b_1 \Delta x + b_0\right)$$

The second finite difference is computed as follows:

$$\Delta^2 f = 2b_2(\Delta x)^2 \qquad (16)$$

# Finite Difference Interpolation



- First and Second Finite Difference
  - The a quadratic polynomial of the form

$$f(x) = b_2 x^2 + b_1 x + b_0$$

  - The first and second finite difference are given as

$$\Delta f = 2b_2(\Delta x)x + \left(b_2(\Delta x)^2 + b_1 \Delta x\right) \qquad (17)$$

and

$$\Delta^2 f = 2b_2(\Delta x)^2 \qquad (18)$$



# Finite Difference Interpolation

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- Example 2
  - Develop an interpolation polynomial for the following data using the finite difference approach. Estimate the $f(x)$ for $x = 2.7$

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(x)$ | 3 | 5 | 8 |

$$\Delta x = 2 - 1 = 3 - 2 = 1$$

# Finite Difference Interpolation



■ Example 2 (cont'd):

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(x)$ | 3 | 5 | 8 |

$\Delta x = 1$

$f(x) = 3$

$f(x + \Delta x) = 5$



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■ Example 2 (cont'd):

– The first finite difference can be obtained using Eq. 1 as

$$\Delta f = f(x + \Delta x) - f(x) = 5 - 3 = 2$$

– This value can be equated to Eq. 17 as follows:

$$\Delta f = 2b_2(\Delta x)x + \left(b_2(\Delta x)^2 + b_1\Delta x\right)$$

$$2 = 2b_2(1)(1) + \left(b_2(1)^2 + b_1(1)\right)$$

$$2 = 3b_2 + b_1 \qquad (19)$$

# Finite Difference Interpolation



- Example 2 (cont'd):
  - The second finite difference can be obtained using Eq. 2 as

$$\Delta^2 f = [f(x+2\Delta x) - f(x+\Delta x)] - [f(x+\Delta x) - f(x)]$$
$$= [8-5] - [5-3] = 1$$

  - This value can be equated to Eq. 18 as follows:

$$\Delta^2 f = 2b_2 (\Delta x)^2$$
$$1 = 2b_2 (1)^2 = 2b_2 \qquad (20)$$



# Finite Difference Interpolation



- Example 2 (cont'd):
  - Solving Eq. 20, gives $b_2 = 0.5$
  - Substituting this result into Eq. 19, gives

$$2 = 3b_2 + b_1 \Rightarrow b_1 = 2 - 3b_2 = 2 - 3(0.5)$$

Or

$$b_1 = 0.5$$

# Finite Difference Interpolation



- Example 2 (cont'd):
  - Using these values of $b_1$ and $b_2$ and any one of the three points, a value of $b_0$ can be calculated from Eq. 12 as follows:

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$b_0 = f(x) = 8 - 0.5(3)^2 - 0.5(3)$$

$$= 2$$


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# Finite Difference Interpolation



- Example 2 (cont'd):
  - Thus, the interpolation polynomial is

$$f(x) = b_2 x^2 + b_1 x + b_0$$

$$f(x) = 0.5x^2 + 0.5x + 2$$

  - $f(2.7)$ can be estimated:

$$f(2.7) = 0.5(2.7)^2 + 0.5(2.7) + 2 = 6.995$$


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# Finite Difference Interpolation



- Finite-difference Tables
  - We see in the preceding example that the finite difference approach can be used to construct an interpolation polynomial.
  - However, the procedure can be tedious and requires a lot of computations.
  - Therefore, we need some sort of scheme to organize the procedure using finite-difference tables.


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# Finite Difference Interpolation

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- Finite-difference Table
  - The finite-differences can be organized into a table as shown in the next viewgraph.
  - The finite-difference table can be used to derive an interpolation polynomial.
  - As we will see, finite-difference tables can be used with Newton formula to derive interpolation polynomials.

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Finite-difference Table

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | … | $\Delta^n f$ |
|---|---|---|---|---|---|---|
| $x$ | $f(x)$ | | | | | |
| | | $\Delta f(x)$ | | | | |
| $x + \Delta x$ | $f(x + \Delta x)$ | | $\Delta^2 f(x)$ | | | |
| | | $\Delta f(x + \Delta x)$ | | $\Delta^3 f(x)$ | | |
| $x + 2\Delta x$ | $f(x + 2\Delta x)$ | | $\Delta^2 f(x + \Delta x)$ | | | |
| | | $\Delta f(x + 2\Delta x)$ | | $\Delta^3 f(x + \Delta x)$ | | |
| $x + 3\Delta x$ | $f(x + 3\Delta x)$ | | $\Delta^2 f(x + 2\Delta x)$ | | | |
| | | $\Delta f(x + 3\Delta x)$ | | $\Delta^3 f(x + 2\Delta x)$ | | |
| | | | $\Delta^2 f(x + 3\Delta x)$ | | | |
| : | : | : | : | : | ::: | $\Delta^n f(x)$ |
| $x + (n-2)\Delta x$ | $f[x + (n-2)\Delta x]$ | | $\Delta^2 f[x + (n-3)\Delta x]$ | | | |
| | | $\Delta f[x + (n-2)\Delta x]$ | | $\Delta^3 f[x + (n-3)\Delta x]$ | | |
| $x + (n-1)\Delta x$ | $f[x + (n-1)\Delta x]$ | | $\Delta^2 f[x + (n-2)\Delta x]$ | | | |
| | | $\Delta f[x + (n-1)\Delta x]$ | | | | |
| $x + n\Delta x$ | $f(x + n\Delta x)$ | | | | | |

# Finite Difference Interpolation

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- Example 3

  Construct a finite-difference table for the following set of data:

| $x$ | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| $f(x)$ | 1000 | 1331 | 1728 | 2197 | 2744 |


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# Finite Difference Interpolation



■ Example 3 (cont'd):

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ |
|---|---|---|---|---|---|
| 10 | 1000 | | | | |
| | | 331 | | | |
| 11 | 1331 | | 66 | | |
| | | 397 | | 6 | |
| 12 | 1728 | | 72 | | 0 |
| | | 469 | | 6 | |
| 13 | 2197 | | 78 | | |
| | | 547 | | | |
| 14 | 2744 | | | | |

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