CHAPTER 6a. NUMERICAL INTERPOLATION

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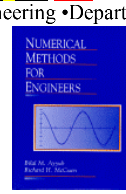

by

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ENCE 203 - Computation Methods in Civil Engineering II

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Introduction

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- Interpolation is a process generally used to estimate a missing functional value by taking a weighted average of known functional values at neighboring points.
- When dealing with tabular values, three basic problems are encountered.

Introduction

- Types of Problems for Tabular Values
  - FIRST:

    Given a mathematical relationship in tabular form, one may wish to extend its range beyond that given by the original data points.
  - SECOND:

    The intention may be to approximate a functional value between two data points

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- Types of Problems for Tabular Values
  - THIRD:

    One may wish to approximate an independent variable corresponding to a given functional value.

Introduction

- Examples Tabular Values
  - Shear Stress of Oil between Two Parallel Plates

Table 1

$$\tau = \mu \frac{V}{h} \quad (1)$$

$$= \mu \frac{\text{velocity}}{\text{gap between plates}}$$

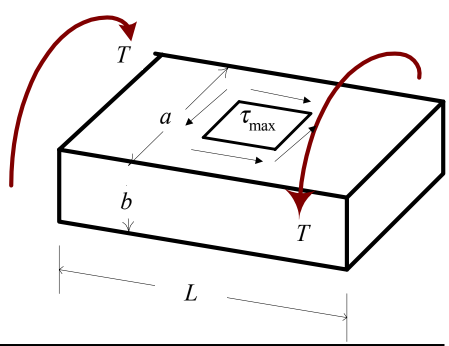
Temperature T (°C)	Viscosity μ (N-sec/m²)
5	0.080
20	0.015
30	0.009
50	0.006
55	0.005

Introduction

- Examples Tabular Values
  - Shearing Stress on Noncircular Element Due to Torque *T*

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$\phi = \frac{TL}{c_2 a b^3 G} \quad (2)$$



Introduction

■ Examples Tabular Values

Table 2

Shearing Stress on Noncircular Element Due to Torque *T*

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$\phi = \frac{TL}{c_2 a b^3 G}$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Introduction

- We see that interpolation is required in many engineering applications that use tabular data as input.
- The basis of all interpolation procedures is fitting of some type curve or function to a sunset of tabular data.
- Interpolation algorithms differ in the form of their interpolation functions.

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■ Example 1: Crude Estimate

Suppose that we want to find the shear stress between two plates for a temperature of 35°C using Eq. 1.

Looking into Table 1, in this case the table does not provide a value for *T* = 35°C.

In this case, we resort to interpolation.

A crude estimate of the viscosity would be to take the average between 2 data points

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■ Example 1 (cont'd): Crude Estimate

Suppose :

$V = 2$ m/sec

$h = 0.025$ m

Temperature T (°C)	Viscosity μ (N-sec/m²)
5	0.080
20	0.015
30	0.009
50	0.006
55	0.005

$$\mu_{\text{average}} = \frac{0.009 + 0.006}{2} = 0.0075 \Rightarrow \tau = \mu \frac{V}{h} = 0.0075 \frac{2}{0.025} = 0.6 \text{ N/m}^2$$

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■ Example 2: Better Estimate

Repeat Example 1 using linear interpolation.

$T(^{o}C)$	$\mu(\text{N-sec/m}^2)$
30	0.009
35	μ_{35}
50	0.006

$$\Rightarrow \frac{35-30}{50-30} = \frac{\mu_{35} - 0.009}{0.006 - 0.009} \Rightarrow \mu_{35} = 0.00825$$

$$\text{Therefore, } \tau = \mu\frac{V}{h} = 0.00825\frac{2}{0.025} = 0.66 \text{ N/m}^2$$

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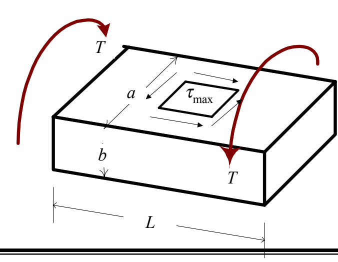
■ Example 3

Use both the crude (average) and linear interpolation methods to find the largest torque which may be applied to the noncircular brass bar as shown. Assume τ = 40× 10^6 Pa, *a* = 0.064 mm, and *b* = 0.025 mm.

$$\tau_{max} = \frac{T}{c_1 ab^2}$$



Introduction

■ Example 3 (cont'd)

Crude estimate:

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56$$

$$c_{1\text{-average}} = \frac{0.258 + 0.267}{2} = 0.2625$$

$$\tau_{\max} = \frac{T}{c_1 ab^2}$$

$$40\times10^6 = \frac{T}{0.2625(0.064)(0.025)^2} \Rightarrow T = 420 \text{ N.m}$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

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■ Example 3 (cont'd)

Linear estimate:

a/b	c_1	c_2
2.5	0.258	0.249
3.0	0.267	0.263

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56$$

a/b	c_1
2.5	0.258
2.56	c_{1_0}
3.0	0.267

$$\Rightarrow \frac{2.56 - 2.5}{3.0 - 2.5} = \frac{c_{1_0} - 0.258}{0.267 - 0.258} \Rightarrow c_{1_0} = 0.2591$$

Introduction

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■ Example 3 (cont'd)

Crude estimate:

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56$$

$$c_{1_0} = 0.2591$$

$$\tau_{max} = \frac{T}{c_1 ab^2}$$

$$40\times10^6 = \frac{T}{0.2591(0.064)(0.025)^2} \Rightarrow T = 414.6 \text{ N.m}$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
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■ Linear Interpolation

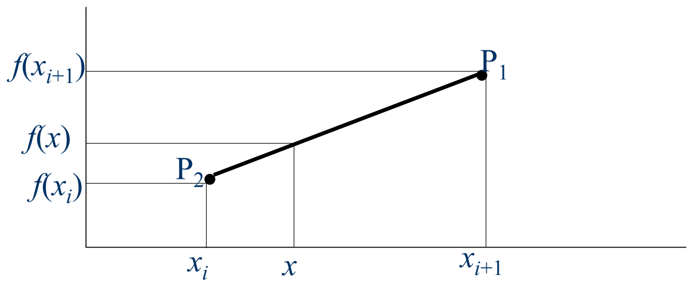
– Linear interpolation is the process of estimating a missing functional value between two data points by constructing a line between these two points.

– The linear function of the line then can be used to find the desired value of the data point of interest.

– In the previous examples, linear interpolation was implicitly employed.

Introduction

- Linear Interpolation
  – Consider the following line connecting two points P_1 and P_2:



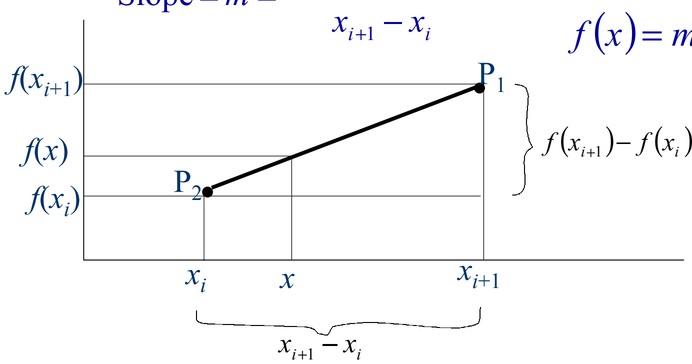
Introduction

- Linear Interpolation

$$\text{Slope} = m = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Equation of a Line:

$$f(x) = mx + c$$



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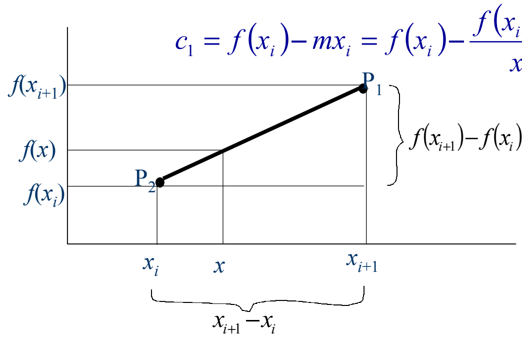
■ Linear Interpolation

Using Point P_2 :

$$f(x_i) = mx_i + c_1$$

$$c_1 = f(x_i) - mx_i = f(x_i) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} x_i$$



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■ Linear Interpolation

$$f(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} x + f(x_i) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} x_i$$

$$f(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} [x - x_i] + f(x_i)$$

or

$$f(x) = f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} [f(x_{i+1}) - f(x_i)]$$

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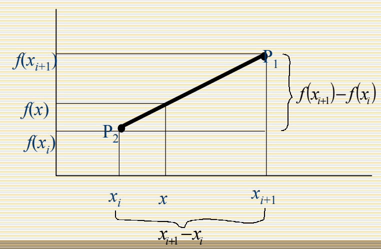
■ Linear Interpolation Function

The linear interpolation function is given by

$$f(x) = f(x_i) + \frac{x - x_i}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right] \qquad (3)$$



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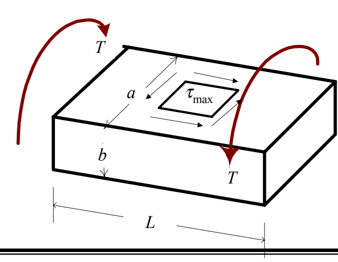
■ Example 4

Use Eq. 3 of linear interpolation function to find the largest torque which may be applied to the noncircular brass bar as shown. Assume $\tau = 40\times 10^6$ Pa, a = 0.064 mm, and b = 0.025 mm.

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$



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■ Example 4 (cont'd)

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56$$

$x = 2.56,\ x_i = 2.5$, and $x_{i+1} = 3$

$f(x_i) = 0.258,\ f(x_{i+1}) = 0.267$, and $f(x)$?

$$c_{1_0} = f(x) = f(x_i) + \frac{x - x_i}{x_{i+1} - x_i}\left[f(x_{i+1}) - f(x_i)\right]$$

$$= 0.258 + \frac{2.56 - 2.5}{3 - 2.5}\left[0.267 - 0.258\right] = 0.2591$$

$$\tau_{\max} = \frac{T}{c_1 ab^2}$$

$$40 \times 10^6 = \frac{T}{0.2591(0.064)(0.025)^2} \Rightarrow T = 414.6 \text{ N.m}$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Method of Undetermined Coefficients

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- The method of undetermined coefficients is conceptually the simplest interpolation algorithm.
- It illustrates many of the key points that hold for all interpolation procedures.
- In this method, an nth-order polynomial is used as the interpolation function $f(x)$.

Method of Undetermined Coefficients

- Interpolation Polynomial
  - The general form of this interpolation polynomial can be given as

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n \quad (4)$$

  - where $b_0, b_1, b_3, \ldots, b_n$ are determined using the measured data points.

Method of Undetermined Coefficients

- Interpolation Polynomial
  - The degree of the polynomial gives the type of the interpolation scheme.
  - For example, a *first-order* interpolation polynomial is actually a linear function of a straight line between two data points, that is

$$f(x) = b_0 + b_1 x \quad (5)$$

Method of Undetermined Coefficients

- Interpolation Polynomial
  - A *second-order* or quadratic interpolation would be given as

$$f(x) = b_0 + b_1 x + b_2 x^2 \qquad (5)$$

  - A *fifth-order* interpolation would be given as

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5$$

Method of Undetermined Coefficients

- Interpolation Polynomial
  - Note that in the linear interpolation polynomial of Eq. 5, the slope of the line is given by the parameter b_1 and the intercept of the line is given by the coefficient b_0.

$$f(x) = b_0 + b_1 x$$

$$f(x) = mx + c$$

Method of Undetermined Coefficients

- Interpolation Polynomial
  - Generally, as the degree of the interpolation polynomial increases, the interpolation function provides more accuracy.
  - Using a function that is more flexible than a straight line, the accuracy of an estimate can be improved.

Method of Undetermined Coefficients

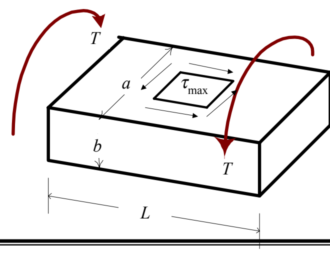
- Interpolation Polynomial
  - A nonlinear function that has more coefficients than the linear interpolation function should provide greater accuracy.
  - However, since a nonlinear function may contain more unknown coefficients than the straight line, it may be necessary to use more data points to estimate the coefficients.

Method of Undetermined Coefficients

■ Example 5

Develop both a linear and quadratic interpolation polynomials for finding the largest torque which may be applied to the noncircular brass bar as shown. Assume $\tau = 40\times 10^6$ Pa, $a = 0.064$ mm, and $b = 0.025$ mm.

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$



Method of Undetermined Coefficients

■ Example 5 (cont'd)

Linear Polynomial

$$\frac{a}{b} = \frac{0.064}{0.025} = 2.56$$

$$f(x) = b_0 + b_1 x$$

$$0.258 = b_0 + b_1(2.5)$$

$$0.267 = b_0 + b_1(3.0)$$

or

$$b_0 + 2.5b_1 = 0.258$$

$$b_0 + \quad 3b_1 = 0.267$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Method of Undetermined Coefficients

■ Example 5 (cont'd)

- We notice that we have a system of two linear equations, which can be solved for the unknowns b_0 and b_1 as follows:

$$\begin{bmatrix} 1 & 2.5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.258 \\ 0.267 \end{bmatrix}$$

Method of Undetermined Coefficients

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■ Example 5 (cont'd)

$$b_0 = \frac{|A_0|}{|A|} = \frac{\begin{vmatrix} 0.258 & 2.5 \\ 0.267 & 3 \end{vmatrix}}{\begin{bmatrix} 1 & 2.5 \\ 1 & 3 \end{bmatrix}} = \frac{0.258(3) - 2.5(0.267)}{(1)(3) - 2.5(1)} = 0.213$$

$$b_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 0.258 \\ 1 & 0.267 \end{vmatrix}}{\begin{bmatrix} 1 & 2.5 \\ 1 & 3 \end{bmatrix}} = \frac{0.267(1) - (1)(0.258)}{(1)(3) - 2.5(1)} = 0.018$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)
  - The linear interpolation function, therefore, is given by

$$f(x) = 0.213 + 0.018x,\ \text{or}$$
$$f(c_1) = 0.213 + 0.018(a/b)$$

  - And the estimate for c_1 at a/b =2.56 will be estimated as

$$f(2.56) = 0.213 + 0.018(2.56) = 0.2591$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$40 \times 10^6 = \frac{T}{0.2591(0.064)(0.025)^2} \Rightarrow T = 414.6 \text{ N.m}$$

Method of Undetermined Coefficients

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■ Example 5 (cont'd)

Quadratic Polynomial

$$f(x) = b_0 + b_1 x + b_2 x^2$$

$$0.246 = b_0 + b_1(2.0) + b_2(2)^2$$

$$0.258 = b_0 + b_1(2.5) + b_2(2.5)^2$$

$$0.267 = b_0 + b_1(3.0) + b_2(3.0)^2$$

or

$$b_0 + 2b_1 + 4b_2 = 0.246$$

$$b_0 + 2.5b_1 + 6.25b_2 = 0.258$$

$$b_0 + 3b_1 + 9b_2 = 0.267$$

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Method of Undetermined Coefficients

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■ Example 5 (cont'd)

- We notice that we have a system of three linear equations, which can be solved for the unknowns b_0, b_1 and b_2 as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.246 \\ 0.258 \\ 0.267 \end{bmatrix}$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.246 \\ 0.258 \\ 0.267 \end{bmatrix}$$

$$b_0 = \frac{|A_0|}{|A|} = \frac{\begin{vmatrix} 0.246 & 2 & 4 \\ 0.258 & 2.5 & 6.25 \\ 0.267 & 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{0.042}{0.25} = 0.168$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.246 \\ 0.258 \\ 0.267 \end{bmatrix}$$

$$b_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 0.246 & 4 \\ 1 & 0.258 & 6.25 \\ 1 & 0.267 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{0.01275}{0.25} = 0.051$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.246 \\ 0.258 \\ 0.267 \end{bmatrix}$$

$$b_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 0.246 \\ 1 & 2.5 & 0.258 \\ 1 & 3 & 0.267 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 1 & 2.5 & 6.25 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{-0.0015}{0.25} = -0.006$$

Method of Undetermined Coefficients

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- Example 5 (cont'd)
  - The quadratic interpolation function, therefore, is given by

$$f(x) = 0.168 + 0.051x - 0.006x^2, \text{ or}$$
$$f(c_1) = 0.168 + 0.051(a/b) - 0.006(a/b)^2$$

  - And the estimate for c_1 at a/b =2.56 will be estimated as

$$f(2.56) = 0.168 + 0.051(2.56) - 0.006(2.56)^2 = 0.25924$$

Method of Undetermined Coefficients

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■ Example 5 (cont'd)

$$\tau_{\max} = \frac{T}{c_1 a b^2}$$

$$40 \times 10^6 = \frac{T}{0.25924(0.064)(0.025)^2} \Rightarrow T = 414.8 \text{ N.m}$$