# CHAPTER 5e. SIMULTANEOUS LINEAR EQUATIONS

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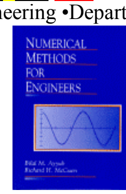


by

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# LU Decomposition

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- A system of simultaneous linear equations can be presented as

$$\begin{array}{ccccccccccc}
a_{11}X_1 &+& a_{12}X_2 &+& a_{13}X_3 &+\cdots+& a_{1n}X_n &=& C_1 \\
a_{21}X_1 &+& a_{22}X_2 &+& a_{23}X_3 &+\cdots+& a_{2n}X_n &=& C_2 \\
a_{31}X_1 &+& a_{32}X_2 &+& a_{33}X_3 &+\cdots+& a_{3n}X_n &=& C_3 \\
\vdots && \vdots && \vdots && \vdots && \vdots \\
a_{n1}X_1 &+& a_{n2}X_2 &+& a_{n3}X_3 &+\cdots+& a_{nn}X_n &=& C_n
\end{array} \qquad (1)$$

# LU Decomposition



- This system of equations can be put in a matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} \qquad (2)$$





# LU Decomposition



- In which, the coefficient matrix ***A*** is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n} \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix}, \text{and } C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}$$

***X*** = vector of unknowns, and ***C*** = vector of constants

# LU Decomposition

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- This system of equations can also be expressed in a compact form as augmented matrix as

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & C_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & C_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & C_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & C_n \end{array}\right] \quad (3)$$



# LU Decomposition

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- The elements of ***L*** and ***B*** can be computed from the elements of ***A*** as follows:

$$l_{i1} = a_{i1} \quad \text{for } i = 1,2,\ldots,n \quad (4a)$$

$$u_{1j} = \frac{a_{1j}}{l_{11}} \quad \text{for } j = 2,3,\ldots,n \quad (4b)$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} \quad \text{for } j = 2,3,\ldots,n-1 \text{ and } i = j, j+1,\ldots,n \quad (4c)$$

$$u_{ji} = \frac{a_{ji} - \sum_{k=1}^{j-1} l_{jk}u_{ki}}{l_{jj}} \quad \text{for } j = 2,3,\ldots,n-1 \text{ and } i = j+1, j+2,\ldots,n \quad (4d)$$

$$l_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn} \quad (4e)$$

# LU Decomposition



- Example 1

For the following system of linear equations, decompose the coefficient matrix into Lower and upper triangular matrices ***L*** and ***U***.

$$X_1 + 3X_2 + 2X_3 = 15$$
$$2X_1 + 4X_2 + 3X_3 = 22$$
$$3X_1 + 4X_2 + 7X_3 = 39$$





# LU Decomposition

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- Example 1 (cont'd)

This system can be expressed in a matrix form as

$$\underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{\boldsymbol{X}} = \underbrace{\begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}}_{\boldsymbol{C}}$$

# LU Decomposition

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■ Example 1 (cont'd)

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix}$$

Using Eq. 4a:

$$l_{i1} = a_{i1}$$

$$i = 1,2 \text{ and } 3, \text{ therefore}$$

$$l_{11} = a_{11} = 1$$

$$l_{21} = a_{21} = 2$$

$$l_{31} = a_{31} = 3$$



# LU Decomposition

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■ Example 1 (cont'd)

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix}$$

Using Eq. 4b:

$$u_{1j} = \frac{a_{1j}}{l_{11}}$$

$$j = 2 \text{ and } 3, \text{ therefore}$$

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{3}{1} = 3$$

$$u_{13} = \frac{a_{13}}{l_{11}} = \frac{2}{1} = 2$$

# LU Decomposition

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■ Example 1 (cont'd)

Using Eq. 4c:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \qquad l_{21} = 2 \quad l_{31} = 3 \quad u_{12} = 3$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}$$

$j = 2$, and $i = 2$ and $3$, therefore

$$l_{22} = a_{22} - \sum_{k=1}^{2-1} l_{ik} u_{kj} = a_{22} - \sum_{k=1}^{1} l_{ik} u_{kj} = a_{22} - l_{21} u_{12} = 4 - (2)(3) = -2$$

$$l_{32} = a_{32} - \sum_{k=1}^{2-1} l_{ik} u_{kj} = a_{32} - \sum_{k=1}^{1} l_{ik} u_{kj} = a_{32} - l_{31} u_{12} = 4 - (3)(3) = -5$$



# LU Decomposition


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■ Example 1 (cont'd)

Using Eq. 4d:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \qquad l_{21} = 2 \quad l_{22} = -2 \quad u_{13} = 2$$

$$u_{ji} = \frac{a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki}}{l_{jj}}$$

$j = 2$, and $i = 3$, therefore

$$u_{23} = \frac{a_{23} - \sum_{k=1}^{2-1} l_{2k} u_{k3}}{l_{22}} = \frac{a_{23} - \sum_{k=1}^{1} l_{2k} u_{k3}}{l_{22}} = \frac{a_{23} - l_{21} u_{13}}{l_{22}} = \frac{3 - 2(2)}{-2} = 0.5$$

# LU Decomposition



■ Example 1 (cont'd)

Using Eq. 4e:

$$l_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk} u_{kn}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \quad \begin{aligned} l_{31} &= 3 \\ l_{32} &= -5 \\ u_{13} &= 2 \\ u_{23} &= 0.5 \end{aligned}$$

Therefore,

$$l_{33} = a_{33} - \sum_{k=1}^{3-1} l_{3k} u_{k3} = a_{33} - \sum_{k=1}^{2} l_{3k} u_{k3}$$

$$= a_{33} - [l_{31} u_{13} + l_{32} u_{23}]$$

$$= 7 - (3)(2) - (-5)(0.5) = 3.5$$



# LU Decomposition



■ Example 1 (cont'd)

– Using Eq. 4, the elements of ***L*** and ***U*** were found to be

$$\begin{aligned} l_{11} &= 1 \\ l_{21} &= 2 \\ l_{31} &= 3 \\ l_{22} &= -2 \\ l_{32} &= -5 \\ l_{33} &= 3.5 \end{aligned} \qquad \begin{aligned} u_{12} &= 3 \\ u_{13} &= 2 \\ u_{23} &= 0.5 \end{aligned}$$

# LU Decomposition



- Example 1 (cont'd)
  - Hence, the lower triangular matrix ***L*** resulting from ***A*** is:

$$l_{11} = 1$$
$$l_{21} = 2$$
$$l_{31} = 3$$
$$l_{22} = -2$$
$$l_{32} = -5$$
$$l_{33} = 3.5$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 3.5 \end{bmatrix}$$





# LU Decomposition



- Example 1 (cont'd)
  - And the upper triangular matrix ***U*** resulting from ***A*** is:

$$u_{12} = 3$$
$$u_{13} = 2$$
$$u_{23} = 0.5$$

$$U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

# LU Decomposition



- Example 2

  For the upper and lower triangular matrices ***U*** and ***L*** computed in Example 1, show that the multiplication of ***LU*** will result in the coefficient matrix ***A***.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 3.5 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix}$$



# LU Decomposition

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- Example 2 (cont'd)
  - Performing the matrix multiplication, gives

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 3.5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2(3)+(-2)(1) & 2(2)+(-2)(0.5) \\ 3 & 3(3)+(-5)(1) & 3(2)+(-5)(0.5)+(3.5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \longleftarrow \boldsymbol{A}$$

# LU Decomposition



- Once the coefficient matrix is decomposed into ***L*** and ***U***, these matrices can be used to solve a system of equations with constants $C_i$, $i = 1,2, \ldots, n$.
- The solution can be obtained in two steps, forward pass and back substitution.




# LU Decomposition



- Recall that the matrix form for the simultaneous linear equations is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} \qquad (5)$$

# LU Decomposition



- Or in terms of the lower and upper triangular matrices ***L*** and ***U*,** the system of equations can be represented by

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_4 \end{bmatrix} \qquad (6)$$



# LU Decomposition

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- Recall the forward pass of the Gaussian elimination that resulted in the following upper triangular matrix:

$$\begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix} \qquad (7)$$

# LU Decomposition

- Forward Pass

  The forward pass produces the $e_i$ values for $i$ =1,2,…, $n$ of equation 7 as follows:

$$e_1 = \frac{C_1}{l_{11}} \qquad (8a)$$

$$e_i = \frac{C_i - \sum_{j=1}^{i-1} l_{ij} e_j}{l_{ii}} \qquad \text{for } i = 2,3,\ldots,n \qquad (8b)$$



# LU Decomposition

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- Back Substitution
  - The back substitution results in the $X_i$ values for $i$ = 1,2,…, $n$ as follows:

$$X_n = e_n \qquad (9a)$$

$$X_i = e_i - \sum_{j=i+1}^{n} u_{ij} X_j \qquad \text{for } i = n-1, n-2, \ldots, 1 \quad (9b)$$

# LU Decomposition



■ Example 3

Solve the system of equations of Example 1 using the LU decomposition approach

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The system of equation in a matrix from is

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$




# LU Decomposition



■ Example 3 (cont'd)

The decomposed lower and upper triangular matrices were found in Example 1 as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & -5 & 3.5 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

# LU Decomposition



■ Example 3 (cont'd)

Using Eq. 8a:

$$e_1 = \frac{C_1}{l_{11}}$$

Therefore,

$$e_1 = \frac{15}{1} = 15$$

$l_{11} = 1 \quad u_{12} = 3$

$l_{21} = 2 \quad u_{13} = 2$

$l_{31} = 3 \quad u_{23} = 0.5$

$l_{22} = -2$

$l_{32} = -5$

$l_{33} = 3.5$

$$C = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$



# LU Decomposition



■ Example 3 (cont'd)

Using Eq. 8b:

$$e_i = \frac{C_i - \sum_{j=1}^{i-1} l_{ij} e_j}{l_{ii}}$$

$i = 2$ and 3, therefore,

$l_{11} = 1 \quad u_{12} = 3$

$l_{21} = 2 \quad u_{13} = 2$

$l_{31} = 3 \quad u_{23} = 0.5$

$l_{22} = -2$

$l_{32} = -5$

$l_{33} = 3.5$

$$C = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$

$$e_2 = \frac{C_2 - \sum_{j=1}^{2-1} l_{2j} e_j}{l_{22}} = \frac{C_2 - \sum_{j=1}^{1} l_{2j} e_j}{l_{22}} = \frac{22 - l_{21} e_1}{l_{22}} = \frac{22 - (2)(15)}{-2} = 4$$

$$e_3 = \frac{C_3 - \sum_{j=1}^{3-1} l_{3j} e_j}{l_{33}} = \frac{C_3 - \sum_{j=1}^{2} l_{3j} e_j}{l_{33}} = \frac{C_3 - l_{31} e_1 - l_{32} e_2}{l_{33}} = \frac{39 - 3(15) - (-5)(4)}{3.5} = 4$$

# LU Decomposition

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■ Example 3 (cont'd)

Using Eq. 9a:

$$X_n = e_n$$

Therefore,

$$X_3 = 4$$

$l_{11} = 1 \quad u_{12} = 3$
$l_{21} = 2 \quad u_{13} = 2$
$l_{31} = 3 \quad u_{23} = 0.5$
$l_{22} = -2$
$l_{32} = -5$
$l_{33} = 3.5$

$$C = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$

$e_1 = 15$
$e_2 = 4$
$e_3 = 4$



# LU Decomposition

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■ Example 3 (cont'd)

Using Eq. 9b:

$$X_i = e_i - \sum_{j=i+1}^{n} u_{ij} X_j$$

$i = 2$ and $1$, therefore,

$e_1 = 15$
$e_2 = 4$
$e_3 = 4$
$X_3 = 4$

$l_{11} = 1 \quad u_{12} = 3$
$l_{21} = 2 \quad u_{13} = 2$
$l_{31} = 3 \quad u_{23} = 0.5$
$l_{22} = -2$
$l_{32} = -5$
$l_{33} = 3.5$

$$C = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$

$$X_2 = e_2 - \sum_{2+1}^{3} u_{2j} X_j = e_2 - u_{23} X_3 = 4 - 0.5(4) = 2$$

$$X_1 = e_1 - \sum_{1+1}^{3} u_{1j} X_j = e_1 - u_{12} X_2 - u_{13} X_3 = 15 - (3)(2) - (2)(4) = 1$$

# LU Decomposition



- Example 3 (cont'd)
  - Therefore, the solution for the system of equations is

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$