CHAPTER 5d. SIMULTANEOUS LINEAR EQUATIONS

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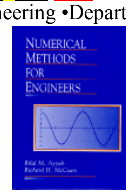

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ENCE 203 - Computation Methods in Civil Engineering II

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Gauss-Jordan Elimination

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- We saw that the Gaussian Elimination consists of two steps:
  - Forward Pass
  - Back Substitution

  For the following set of equation:

$$\lfloor a_{ij} \rceil \{X_i\} = \{C_i\} \qquad (1)$$

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Gauss-Jordan Elimination

- The *Forward Pass* can always result into a augmented matrix in the general form as

$$\begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix} \quad (2)$$

Gauss-Jordan Elimination

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- The *Back Substitution* can always result into the following general form:

$$\begin{aligned} X_n &= e_n \\ X_{n-1} &= e_{n-1} - d_{n-1,n} X_n \\ X_{n-2} &= e_{n-2} - d_{n-2,n-1} X_{n-1} - d_{n-2,n} X_n \\ &\vdots \\ X_1 &= e_1 - d_{12} X_2 - d_{13} X_3 - \cdots - d_{1,n} X_n \end{aligned} \quad (3)$$

Gauss-Jordan Elimination

- Gauss-Jordan Process
  - An alternative process of elimination in which all coefficients in a column except the pivot element are eliminated can also be used to obtain a solution.
  - In Gauss-Jordan elimination, the solution is obtained directly after performing the forward pass.
  - There is no *back substitution*.

Gauss-Jordan Elimination

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- Gauss-Jordan Process
  - Given the following set of equation:

$$
\begin{aligned}
a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n &= C_1 \\
a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n &= C_2 \\
a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n &= C_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad & \vdots \qquad \vdots \\
a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n &= C_n
\end{aligned}
\qquad (4)
$$

Gauss-Jordan Elimination

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- Gauss-Jordan Process
  - Or given the following set of equation in augmented matrix of the form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & | & C_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & | & C_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & | & C_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & | & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & | & C_n \end{bmatrix} \qquad (5)$$

Gauss-Jordan Elimination

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- Gauss-Jordan Process
  - Using Gauss-Jordan, Eq. 5 can be transformed into the following form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & | & C_1^* \\ 0 & 1 & 0 & \cdots & 0 & | & C_2^* \\ 0 & 0 & 1 & \cdots & 0 & | & C_3^* \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & 0 & 0 & 1 & | & C_n^* \end{bmatrix} \qquad (6)$$

Gauss-Jordan Elimination

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- Gauss-Jordan Process
  - And in which the solution is readily obtained as

$$\begin{aligned} X_1 &= C_1^* \\ X_2 &= C_2^* \\ X_3 &= C_3^* \\ \vdots \quad & \quad \vdots \\ X_n &= C_n^* \end{aligned} \qquad (7)$$

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Gauss-Jordan Elimination

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- Example: Gauss-Jordan
  - Solve the following set of simultaneous equations using the Gauss-Jordan Method:

$$\begin{aligned} X_1 + 3X_2 + 2X_3 &= 15 \\ 2X_1 + 4X_2 + 3X_3 &= 22 \\ 3X_1 + 4X_2 + 7X_3 &= 39 \end{aligned}$$

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Gauss-Jordan Elimination

■ Example (cont'd): Gauss-Jordan

– This system of equations can be represented in a matrix form as

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix}$$

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Gauss-Jordan Elimination

■ Example (cont'd): Gauss-Jordan

– Or this system of equations can be represented in an augmented matrix form as

$$\begin{bmatrix} 1 & 3 & 2 & | & 15 \\ 2 & 4 & 3 & | & 22 \\ 3 & 4 & 7 & | & 39 \end{bmatrix}$$

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Gauss-Jordan Elimination

- Example (cont'd): Gauss-Jordan
  - Step 1

$$\begin{bmatrix} 1 & 3 & 2 & | & 15 \\ 2 & 4 & 3 & | & 22 \\ 3 & 4 & 7 & | & 39 \end{bmatrix} \begin{matrix} R_1' = R_1 \\ R_2' = R_2 - 2R_1' \\ R_3' = R_3 - 3R_1' \end{matrix} \begin{bmatrix} 1 & 3 & 2 & | & 15 \\ 0 & -2 & -1 & | & -8 \\ 0 & -5 & -1 & | & -6 \end{bmatrix}$$

$$\begin{array}{lcccc} R_2 : 2 & 4 & 3 & 22 & \\ 2R_1' : 2 & 6 & 4 & 30 & (\text{-}) \\ \hline R_2' : 0 & \text{-}2 & \text{-}1 & \text{-}8 & \end{array} \qquad \begin{array}{lcccc} R_3 : 3 & 4 & 7 & 39 & \\ 3R_1' : 3 & 9 & 6 & 45 & (\text{-}) \\ \hline R_3' : 0 & \text{-}5 & 1 & \text{-}6 & \end{array}$$

Gauss-Jordan Elimination

- Example (cont'd): Gauss-Jordan
  - Step 2

$$\begin{bmatrix} 1 & 3 & 2 & | & 15 \\ 0 & -2 & -1 & | & -8 \\ 0 & -5 & 1 & | & -6 \end{bmatrix} \begin{matrix} R_1' = R_1 - 3R_2' \\ R_2' = R_2 / -2 \\ R_3' = R_3 + 5R_2' \end{matrix} \begin{bmatrix} 1 & 0 & 1/2 & | & 3 \\ 0 & 1 & 1/2 & | & 4 \\ 0 & 0 & 7/2 & | & 14 \end{bmatrix}$$

$$\begin{array}{lcccc} R_1 : 1 & 3 & 2 & 15 & \\ 3R_2' : 0 & 3 & 3/2 & 12 & (\text{-}) \\ \hline R_1' : 1 & 0 & 1/2 & 3 & \end{array} \qquad \begin{array}{lcccc} R_3 : 0 & -5 & 1 & -6 & \\ 5R_2' : 0 & 5 & 5/2 & 20 & (+) \\ \hline R_3' : 0 & 0 & 7/2 & 14 & \end{array}$$

Gauss-Jordan Elimination

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- Example (cont'd): Gauss-Jordan
  - Step 3

$$\begin{bmatrix} 1 & 0 & 1/2 & | & 3 \\ 0 & 1 & 1/2 & | & 4 \\ 0 & 0 & 7/2 & | & 14 \end{bmatrix} \begin{matrix} R_1' = R_1 - R_3'/2 \\ R_2' = R_2 - R_3'/2 \\ R_3' = R_3/(7/2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

$$\begin{array}{lllll} R_1 & :1 & 0 & 1/2 & 3 & \\ R_3'/2 & :0 & 0 & 1/2 & 2 & (-) \\ \hline R_1' & :1 & 0 & 0 & 1 & \end{array}$$

$$\begin{array}{lllll} R_2 & :0 & 1 & 1/2 & 4 & \\ R_3'/2 & :0 & 0 & 1/2 & 2 & (-) \\ \hline R_2' & :0 & 1 & 0 & 2 & \end{array}$$

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Gauss-Jordan Elimination

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- Example (cont'd): Gauss-Jordan
  - The last augmented matrix gives the solution for the system of equations as follows:

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

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Gauss-Jordan Elimination

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- Example (cont'd): Gauss-Jordan
  - OR

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 39 \end{bmatrix} \Longrightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

LU Decomposition

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- Recall that the Guassian elimination process consists of two steps:
  - Forward Pass
  - Back Substitution
- The forward pass transforms the system of equations into upper triangular matrix.

LU Decomposition

- The upper triangular matrix can be denoted as ***U*** and it is given by

$$\begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

LU Decomposition

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- Recall the augmented matrix that can result from the forward pass of Gaussian Elimination:

$$\begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix} \qquad (9)$$

LU Decomposition

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- The values of this matrix (Eq. 8) can be related to the values in Eq. 9 as

$$u_{ij} = d_{ij} \quad \text{for } i = 1,2,\cdots,n$$
$$\text{and } j = 1,2,\cdots,n$$

- The upper triangular matrix ***U*** can be related to the coefficients matrix ***A*** as

LU Decomposition

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$$LU = A \qquad (10)$$

Where ***L*** = lower triangular matrix, which can be viewed as to present the back substitution

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \qquad (11)$$

LU Decomposition

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- Therefore Eq. 10 can be expressed as

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \tag{12}$$

$$LU = A$$

LU Decomposition

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- Eq. 12 states that the coefficient matrix ***A*** can be decomposed to two triangular matrices ***L*** and ***U***.
- These triangular matrices can be determined without the use of Gaussian elimination method by performing the matrix multiplication ***LU*** in Eq. 12

LU Decomposition

■ Multiplication of ***LU*** will result in

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \qquad (13)$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & \cdots & l_{11}u_{1n} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & \cdots & l_{21}u_{1n} + l_{22}u_{2n} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} & \cdots & l_{31}u_{1n} + l_{32}u_{n2} + l_{33}u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n1}u_{12} + l_{n2} & l_{n1}u_{13} + l_{n2}u_{23} + l_{n3} & \cdots & l_{n1}u_{1n} + l_{n2}u_{2n} + l_{n3}u_{3n} + \cdots + l_{nn} \end{bmatrix}$$

LU Decomposition

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■ Multiplication of ***LU*** will result in

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & \cdots & l_{11}u_{1n} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & \cdots & l_{21}u_{1n} + l_{22}u_{2n} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} & \cdots & l_{31}u_{1n} + l_{32}u_{n2} + l_{33}u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n1}u_{12} + l_{n2} & l_{n1}u_{13} + l_{n2}u_{23} + l_{n3} & \cdots & l_{n1}u_{1n} + l_{n2}u_{2n} + l_{n3}u_{3n} + \cdots + l_{nn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{31} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \qquad (14)$$

LU Decomposition

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- For example, the multiplication of the first row of ***L*** with the first column of ***U*** results un the following value that is equal to a_{11}.

$$l_{11}(1) = a_{11}$$

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- In view of Eq. 14, for example

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & \cdots & l_{11}u_{1n} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} & \cdots & l_{21}u_{1n}+l_{22}u_{2n} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} & \cdots & l_{31}u_{1n}+l_{32}u_{n2}+l_{33}u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n1}u_{12}+l_{n2} & l_{n1}u_{13}+l_{n2}u_{23}+l_{n3} & \cdots & l_{n1}u_{1n}+l_{n2}u_{2n}+l_{n3}u_{3n}+\cdots+l_{nn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{31} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$l_{31} = a_{31}$$

$$l_{11}u_{13} = a_{31} \Rightarrow u_{13} = \frac{a_{31}}{l_{11}}$$

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LU Decomposition

- From Eq. 14, the values for ***L*** and *U* can be computed from:

$$l_{i1} = a_{i1} \quad \text{for } i = 1,2,\ldots,n$$

$$u_{1j} = \frac{a_{1j}}{l_{11}} \quad \text{for } j = 2,3,\ldots,n$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} \quad \text{for } j = 2,3,\ldots,n-1 \text{ and } i = j, j+1,\ldots,n$$

$$u_{ji} = \frac{a_{ji} - \sum_{k=1}^{j-1} l_{jk}u_{ki}}{l_{jj}} \quad \text{for } j = 2,3,\ldots,n-1 \text{ and } i = j+1, j+2,\ldots,n$$

$$l_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn}$$

(15)

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LU Decomposition

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- Once the coefficient matrix is decomposed into ***L*** and ***U***, these matrices can be used to solve a system of equations with constants C_i, $i = 1,2, \ldots, n$.
- The solution can be obtained in two steps, forward pass and back substitution.

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