CHAPTER 5c. SIMULTANEOUS LINEAR EQUATIONS

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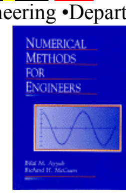

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ENCE 203 - Computation Methods in Civil Engineering II

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Gaussian Elimination

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- Gaussian Elimination Method
  - Earlier we saw how the system of two equation was solved by the elimination of the unknowns:

$$a_{11}X_1 + a_{12}X_2 = C_1 \qquad (1)$$

$$a_{21}X_1 + a_{22}X_2 = C_2 \qquad (2)$$

$$X_1 = \frac{C_1 - a_{12}X_2}{a_{11}} \qquad (3) \qquad\qquad a_{21}\frac{C_1 - a_{12}X_2}{a_{11}} + a_{22}X_2 = C_2 \qquad (4)$$

Gaussian Elimination

■ Gaussian Elimination Method

– Equation 4 is a single equation with one unknown, X_2.

– This equation can be solved for X_2 to give

$$X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} \qquad (5)$$

– Eq. 5 can be substituted back into Eq. 3 to give

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} \qquad (6)$$

Gaussian Elimination

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■ Gaussian Elimination Method

– In the previous example of two-equation systems, the procedure consists of two steps:

- The equation were manipulated to eliminate one of the unknowns from the equations. The result of this *elimination* step was that we had one equation with one unknown.
- Consequently, this equation could be solved directly and the result *back-substituted* into one of the original equations to solve for the rest.

Gaussian Elimination

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- Gaussian Elimination Method
  - This basic approach can be extended to large systems of simultaneous equations by developing a systematic scheme or algorithm to *eliminate* the unknowns, and to *back-substitute*.
  - Gaussian elimination method is the most basic of these schemes.

Gaussian Elimination

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- Gaussian Elimination Procedure

  The Gaussian elimination procedure can be separated into two parts:
  1. Forward Pass
  2. Back Substitution

Gaussian Elimination

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■ Gaussian Elimination Procedure

– Forward Pass

• The process begins with the arrangement of the system of equations in such a manner that $a_{11} = 1$, to give

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n &= c_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n &= c_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n &= c_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n &= c_n \end{aligned} \qquad (7)$$

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– Forward Pass

• Dividing the first equation in (EQ. 7) by a_{11} gives

$$\begin{aligned} X_1 + a'_{12}X_2 + a'_{13}X_3 + \cdots + a'_{1n}X_n &= C'_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n &= C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n &= C_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n &= C_n \end{aligned} \qquad (8)$$

Gaussian Elimination

– Forward Pass

- Multiplying the first equation in (EQ. 8) by $-a_{i1}$ for $i = 2,\ldots,n$ then adding to the ith equation eliminates X_1 from all but the first equation to give

$$\begin{aligned} X_1 + a'_{12}X_2 + a'_{13}X_3 + \cdots + a'_{1n}X_n &= C'_1 \\ a'_{22}X_2 + a'_{23}X_3 + \cdots + a'_{2n}X_n &= C'_2 \\ a'_{32}X_2 + a'_{33}X_3 + \cdots + a'_{3n}X_n &= C'_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ a'_{n2}X_2 + a'_{n3}X_3 + \cdots + a'_{nn}X_n &= C'_n \end{aligned} \qquad (9)$$

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– Forward Pass

- The second equation in (EQ. 9) is now divided by a'_{22} to give

$$\begin{aligned} X_1 + a'_{12}X_2 + a'_{13}X_3 + \cdots + a'_{1n}X_n &= C'_1 \\ X_2 + a''_{23}X_3 + \cdots + a''_{2n}X_n &= C''_2 \\ a'_{32}X_2 + a'_{33}X_3 + \cdots + a'_{3n}X_n &= C'_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ a'_{n2}X_2 + a'_{n3}X_3 + \cdots + a'_{nn}X_n &= C'_n \end{aligned} \qquad (10)$$

Gaussian Elimination

– Forward Pass
  - Multiplying the second equation in (EQ. 10) by $-a'_{i2}$ for $i = 3,\ldots,n$ then adding to the ith equation eliminates x_2 from all but the first and second equations.
  - This process is continued until one equation in one unknown remains.
  - Note that at each stage the remaining equations may require rearranging to avoid a zero divisor in a_{ii} position.

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– Forward Pass
  - Once the process is completed, the system of equations as given by EQ. 7 should have the following triangular form:

$$
\begin{aligned}
X_1 + d_{12}X_2 + d_{13}X_3 + \cdots + d_{1n}X_n &= e_1 \\
X_2 + d_{23}X_3 + \cdots + d_{2n}X_n &= e_2 \\
X_3 + \cdots + d_{3n}X_n &= e_3 \\
\vdots \quad & \quad \vdots \\
X_n &= e_n
\end{aligned}
\qquad (11)
$$

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Forward Pass:

- EQ. 11 can be written in a more compact form, in an augmented matrix as

$$\begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix} \qquad (12)$$

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– Forward Pass

- Hence the original system of equations can be transformed to a triangular matrix form as follows:

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n &= b_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n &= b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \quad \vdots \\ a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n &= b_n \end{aligned} \Longrightarrow \begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix}$$

Gaussian Elimination

■ Example: Forward Pass

Perform a forward pass to transform the following set of equations to a triangular matrix form:

$$\begin{aligned} 3X_1 - 2X_2 + 4X_3 &= 18 \\ X_1 + X_2 - 2X_3 &= 6 \\ 2X_1 + 3X_2 + X_3 &= 10 \end{aligned} \qquad (13)$$

Gaussian Elimination

■ Example (cont'd): Forward Pass

– Dividing the first equation in (EQ. 13) by $a_{11} = 3$, gives

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_1 + X_2 - 2X_3 &= 6 \\ 2X_1 + 3X_2 + X_3 &= 10 \end{aligned} \qquad (14)$$

Gaussian Elimination

- Example (cont'd): Forward Pass
  - Multiplying the first equation in (EQ. 14) by $-a_{21}= -1$ then adding to the 2nd equation eliminates X_1 from the second equation:

$$
\begin{aligned}
-X_1 + \frac{2}{3}X_2 - \frac{4}{3}X_3 &= -6 \\
X_1 + X_2 - 2X_3 &= 6 \\
\hline
\frac{5}{3}X_2 - \frac{10}{3}X_3 &= 0
\end{aligned}
$$

$$
\begin{aligned}
X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\
\frac{5}{3}X_2 - \frac{10}{3}X_3 &= 0 \\
2X_1 + 3X_2 + X_3 &= 10
\end{aligned}
\qquad (15)
$$

Gaussian Elimination

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- Example (cont'd): Forward Pass
  - Multiplying the first equation in (EQ. 14) by $-a_{31}= -2$ then adding to the 3nd equation eliminates X_1 from the third equation:

$$
\begin{aligned}
-2X_1 + \frac{4}{3}X_2 - \frac{8}{3}X_3 &= -12 \\
2X_1 + 3X_2 + X_3 &= 10 \\
\hline
\frac{13}{3}X_2 - \frac{5}{3}X_3 &= -2
\end{aligned}
$$

$$
\begin{aligned}
X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\
\frac{5}{3}X_2 - \frac{10}{3}X_3 &= 0 \\
\frac{13}{3}X_2 - \frac{5}{3}X_3 &= -2
\end{aligned}
\qquad (16)
$$

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■ Example (cont'd): Forward Pass

– The second equation in (EQ. 16) is now divided by $a_{22} = 5/3$ to give

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_2 - 2X_3 &= 0 \\ \frac{13}{3}X_2 - \frac{5}{3}X_3 &= -2 \end{aligned} \qquad (17)$$

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■ Example (cont'd): Forward Pass

– Multiplying the second equation in (EQ. 17) by $-a_{31} = -13/3$ then adding to the 3rd equation eliminates X_2 from the third equation:

$$\begin{aligned} -\frac{13}{3}X_2 + \frac{26}{3}X_3 &= 0 \\ \frac{13}{3}X_2 - \frac{5}{3}X_3 &= -2 \\ \hline 7X_3 &= -2 \end{aligned}$$

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_2 - 2X_3 &= 0 \\ 7X_3 &= -2 \end{aligned} \qquad (18)$$

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■ Example (cont'd): Forward Pass

– The third equation in (EQ. 18) is now divided by $a_{33} = 7$ to give

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_2 - 2X_3 &= 0 \\ X_3 &= -\frac{2}{7} \end{aligned} \qquad (19)$$

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■ Example (cont'd): Forward Pass

– Hence, the required triangular matrix form of the forward pass is

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_2 - 2X_3 &= 0 \\ X_3 &= -\frac{2}{7} \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & -2/3 & 4/3 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2/7 \end{bmatrix}$$

Gaussian Elimination

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- Gaussian Elimination Procedure
  - Back Substitution
    - We saw that the original system of equations was transformed to a triangular matrix form by the forward pass:

$$\begin{matrix} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n = C_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n = C_2 \\ a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \cdots + a_{3n}X_n = C_3 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \cdots + a_{nn}X_n = C_n \end{matrix} \Longrightarrow \begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} & e_1 \\ 0 & 1 & d_{23} & \cdots & d_{2n} & e_2 \\ 0 & 0 & 1 & \cdots & d_{3n} & e_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & e_n \end{bmatrix} \quad (20)$$

Gaussian Elimination

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- Gaussian Elimination Procedure
  - Back Substitution
    - After the forward pass is completed, the unknowns in the equations are found by back substitution procedure.
    - The back substitution procedure can be better illustrated if the matrix of Eq. 20 is written in the equivalent form in terms of individual equations as shown in the next viewgraph.

Gaussian Elimination

- Gaussian Elimination Procedure
  - Back Substitution

$$
\begin{aligned}
X_1 + d_{12}X_2 + d_{13}X_3 \cdots + d_{1,n-2}X_{n-2} + d_{1,n-1}X_{n-1} + d_{1,n}X_n &= e_1 \\
X_2 + d_{23}X_3 \cdots + d_{2,n-2}X_{n-2} + d_{2,n-1}X_{n-1} + d_{2,n}X_n &= e_2 \\
X_3 \cdots + d_{3,n-2}X_{n-2} + d_{3,n-1}X_{n-1} + d_{3,n}X_n &= e_3 \\
X_{n-2} + d_{n-2,n-1}X_{n-1} + d_{n-2,n}X_n &= e_{n-2} \\
X_{n-1} + d_{n-1,n}X_n &= e_{n-1} \\
X_n &= e_n
\end{aligned} \qquad (21)
$$

Gaussian Elimination

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- Gaussian Elimination Procedure
  - Back Substitution
    - From EQS. 21, which represent the system of equations after the forward pass, it is now easy to obtain the solution for X_i. The last equation in EQS. 21 involves only a single unknown; thus the value of is given by

$$X_n = e_n$$

Gaussian Elimination

- Gaussian Elimination Procedure
  - Back Substitution
    - Therefore, the unknowns are determined by back substitution as follows:

$$\begin{aligned}
X_n &= e_n \\
X_{n-1} &= e_{n-1} - d_{n-1,n} X_n \\
X_{n-2} &= e_{n-2} - d_{n-2,n-1} X_{n-1} - d_{n-2,n} X_n \\
&\vdots \\
X_1 &= e_1 - d_{12} X_2 - d_{13} X_3 - \cdots - d_{1,n} X_n
\end{aligned}$$

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Back Substitution

The following set of equations can be used to find the solution by back substitution:

$$\begin{aligned}
X_n &= e_n \\
X_{n-1} &= e_{n-1} - d_{n-1,n} X_n \\
X_{n-2} &= e_{n-2} - d_{n-2,n-1} X_{n-1} - d_{n-2,n} X_n \qquad (22) \\
&\vdots \\
X_1 &= e_1 - d_{12} X_2 - d_{13} X_3 - \cdots - d_{1,n} X_n
\end{aligned}$$

Gaussian Elimination

■ Example: Back Substitution

The forward pass was performed in the last example to transform the following set of equations to a triangular matrix form:

$$\begin{aligned} 3X_1 - 2X_2 + 4X_3 &= 18 \\ X_1 + X_2 - 2X_3 &= 6 \\ 2X_1 + 3X_2 + X_3 &= 10 \end{aligned} \qquad (23)$$

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■ Example (cont'd): Back Substitution

– The forward pass resulted in the following set of equations:

$$\begin{aligned} X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 &= 6 \\ X_2 - 2X_3 &= 0 \\ X_3 &= -\frac{2}{7} \end{aligned} \Longrightarrow \begin{bmatrix} 1 & -2/3 & 4/3 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2/7 \end{bmatrix}$$

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■ Example (cont'd): Back Substitution

– The back substitution will give the solution as follows:

$$X_3 = -\frac{2}{7}$$
$$X_2 - 2X_3 = 0$$
$$X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 = 6$$

$\Longrightarrow$

$$X_3 = -\frac{2}{7}$$
$$X_2 = 2\left(-\frac{2}{7}\right) = -\frac{4}{7}$$
$$X_1 = 6 + \left(\frac{2}{3}\right)\left(-\frac{4}{7}\right) - \left(\frac{4}{3}\right)\left(\frac{-2}{7}\right) = 6$$

Gaussian Elimination

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■ Summary of Gaussian Method

Operation	Symbol
Step 1: Construct the augmented matrix of the $[a_{ij}]$ matrix and $\{C_i\}$ vector	$\lfloor a_{ij} \vdots C_i \rfloor\; i = 1, \cdots, n$ $j = 1, \cdots, n$
Step 2: Check a_{11}; if it is equal to zero then interchange rows so that $a_{11} \neq 0$	

Gaussian Elimination

■ Summary of Gaussian Method

Operation	Symbol
Step 3: Divide row one by a_{11} to get new coefficient a'_{ij} where $a_{11} = 1$	$a'_{ij} = \frac{a_{ij}}{a_{11}}$
Step 4: Multiply row one by $-a_{i1}$ and add to the ith row for $I = 2, \ldots, n$	$-a_{i1}R_1 + R_i,$ $i = 2, \cdots, n$

Gaussian Elimination

■ Summary of Gaussian Method

Operation	Symbol
Step 5: Repeat steps 2, 3, and 4 for the second through $(n-1)^{th}$ rows	
Step 6: Solve for X_n from the nth equation	$X_n = e_n$

Gaussian Elimination

■ Summary of Gaussian Method

Operation	Symbol
Step 7: Solve for $X_{n-1}, \ldots, X_1$	$X_j = e_j - \sum_{r=j+1}^{n} d_{jr} X_r$

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■ Example:

Solve the following set of simultaneous linear equations using the Gaussian method:

$$2X_1 + 3X_2 - 2X_3 - X_4 = -2$$
$$2X_1 + 5X_2 - 3X_3 + X_4 = 7$$
$$-2X_1 + X_2 + 3X_3 - 2X_4 = 1$$
$$-5X_1 + 2X_2 - X_3 + 3X_4 = 8$$

Gaussian Elimination

■ Example (cont'd):

– Forward Pass

Original Matrix	Operation	Resultant Matrix
$\begin{bmatrix} 2 & 3 & -2 & -1 & -2 \\ 2 & 5 & -3 & 1 & 7 \\ -2 & 1 & 3 & -2 & 1 \\ -5 & 2 & -1 & 3 & 8 \end{bmatrix}$	$R_1' = R_1/2$ $R_2' = R_2 - 2R_1'$ $R_3' = R_3 + 2R_1'$ $R_4' = R_4 + 5R_1'$	$\begin{bmatrix} 1 & 2/3 & -1 & -1/2 & -1 \\ 0 & 2 & -1 & 2 & 9 \\ 0 & 4 & 1 & -3 & -1 \\ 0 & 19/2 & -6 & 1/2 & 3 \end{bmatrix}$

In the above notation, the operation column describes the row operations Performed on each row R_i, where R_i = row of vector values and R'_i is the Resulting value.

Gaussian Elimination

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■ Example (cont'd):

– Forward Pass

Original Matrix	Operation	Resultant Matrix
$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 2 & -1 & 2 & 9 \\ 0 & 4 & 1 & -3 & -1 \\ 0 & 19/2 & -6 & 1/2 & 3 \end{bmatrix}$	$R_1' = R_1$ $R_2' = R_2/2$ $R_3' = R_3 - 4R_2'$ $R_4' = R_4 - 19/2(R_2')$	$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 3 & -7 & -19 \\ 0 & 0 & -5/4 & -9 & -159/4 \end{bmatrix}$

Gaussian Elimination

■ Example (cont'd):

– Forward Pass

Original Matrix

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 3 & -7 & -19 \\ 0 & 0 & -5/4 & -9 & -159/4 \end{bmatrix}$$

Operation

$$R_1' = R_1 \quad R_2' = R_2 \quad R_3' = R_3/3 \quad R_4' = R_4 - 5/4(R_3')$$

Resultant Matrix

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & -143/12 & -572/12 \end{bmatrix}$$

Gaussian Elimination

■ Example (cont'd):

– Forward Pass

Original Matrix

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & -143/12 & -159/12 \end{bmatrix}$$

Operation

$$R_1' = R_1 \quad R_2' = R_2 \quad R_3' = R_3 \quad R_4' = R_4/(-143/12)$$

Resultant Matrix

$$\begin{bmatrix} 1 & 3/2 & -1 & -1/2 & -1 \\ 0 & 1 & -1/2 & 1 & 9/2 \\ 0 & 0 & 1 & -7/3 & -19/3 \\ 0 & 0 & 0 & 1 & 572/143 \end{bmatrix}$$

Gaussian Elimination

■ Example (cont'd):

– Forward Pass

The resultant matrix of the last operation represents the following set:

$$X_1 + \frac{3}{2}X_2 - X_3 - \frac{1}{2}X_4 = -1$$

$$X_2 - \frac{1}{2}X_3 + X_4 = \frac{9}{2}$$

$$X_3 - \frac{7}{3}X_4 = -\frac{19}{3}$$

$$X_4 = \frac{572}{143} = 4$$

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■ Example (cont'd):

– Back Substitution

$$X_4 = 4$$

$$X_3 - \frac{7}{3}X_4 = -\frac{19}{3}$$

$$X_2 - \frac{1}{2}X_3 + X_4 = \frac{9}{2}$$

$$X_1 + \frac{3}{2}X_2 - X_3 - \frac{1}{2}X_4 = -1$$

$\Rightarrow$

$$X_4 = 4$$

$$X_3 = -\frac{19}{3} + \frac{7}{3}(4) = 3$$

$$X_2 = \frac{9}{2} + \frac{1}{2}(3) - 4 = 2$$

$$X_1 + \frac{3}{2}X_2 - X_3 - \frac{1}{2}X_4 = -1 - \frac{3}{2}(2) + 3 + \frac{1}{2}(4) = 1$$

$$\therefore X_1 = 1, X_2 = 2, X_3 = 3, \text{ and } X_4 = 4$$