CHAPTER 5b. SIMULTANEOUS LINEAR EQUATIONS

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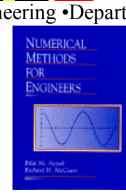

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Simultaneous Linear Equations

- Types of Numerical Procedures:
  1. Elimination methods,
  2. Iteration methods, and
  3. Method of determinants.

Simultaneous Linear Equations

- Classification of Systems of Equations Based on Graphical Interpretation:
  1. Systems that have solutions,
  2. Systems without solution, and
  3. Systems with an infinite number of solutions.

Simultaneous Linear Equations

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- System with a Solution
  - Consider the following two system of simultaneous equation:

$$2X_1 + 3X_2 = 6$$
$$2X_1 + 9X_2 = 12$$

  - This system yields the following solution:

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{9(6) - 3(12)}{2(9) - 2(3)} = 1.5 \qquad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{2(12) - 2(6)}{2(9) - 2(3)} = 1$$

Simultaneous Linear Equations

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■ System with a Solution

Eq.1: $2X_1 + 3X_2 = 6$
Eq.2: $2X_1 + 9X_2 = 12$

$$X_2 = \frac{6 - 2X_1}{3}$$

$$X_2 = \frac{12 - 2X_1}{9}$$

X_1	Eq.1	Eq.2
0.000	2.000	1.333
0.200	1.867	1.289
0.400	1.733	1.244
0.600	1.600	1.200
0.800	1.467	1.156
1.000	1.333	1.111
1.200	1.200	1.067
1.400	1.067	1.022
1.600	0.933	0.978
1.800	0.800	0.933
2.000	0.667	0.889
2.200	0.533	0.844
2.400	0.400	0.800

Simultaneous Linear Equations

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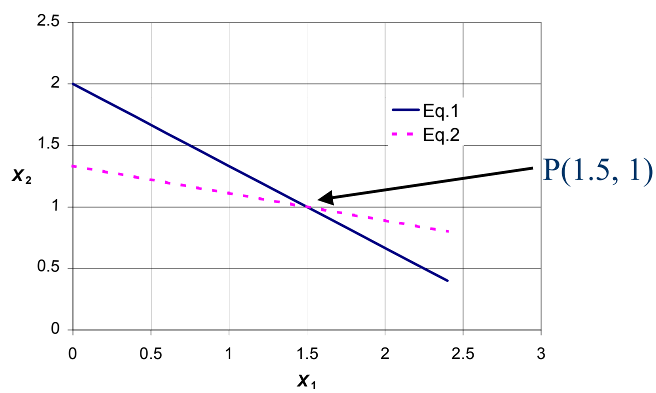
■ System with a Solution

Eq.1: $2X_1 + 3X_2 = 6$
Eq.2: $2X_1 + 9X_2 = 12$



Simultaneous Linear Equations

- System without a Solution
  - Consider the following two system of simultaneous equation:

$$3X_1 + 9X_2 = 5$$
$$X_1 + 3X_2 = 6$$

  - This system does not have a solution

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{3(5) - 9(6)}{3(3) - 9(1)} = \frac{-39}{0} = \infty \quad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{3(6) - 1(5)}{3(3) - 9(1)} = \frac{13}{0} = \infty$$

Simultaneous Linear Equations

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- System without a Solution

$$\text{Eq.1}: 3X_1 + 9X_2 = 5$$
$$\text{Eq.2}: X_1 + 3X_2 = 6$$

$$X_2 = \frac{5 - 3X_1}{9}$$
$$X_2 = \frac{6 - X_1}{3}$$

X_1	Eq.1	Eq.2
0.0	0.556	2.000
0.2	0.489	1.933
0.4	0.422	1.867
0.6	0.356	1.800
0.8	0.289	1.733
1.0	0.222	1.667
1.2	0.156	1.600
1.4	0.089	1.533
1.6	0.022	1.467
1.8	-0.044	1.400
2.0	-0.111	1.333
2.2	-0.178	1.267
2.4	-0.244	1.200

Simultaneous Linear Equations

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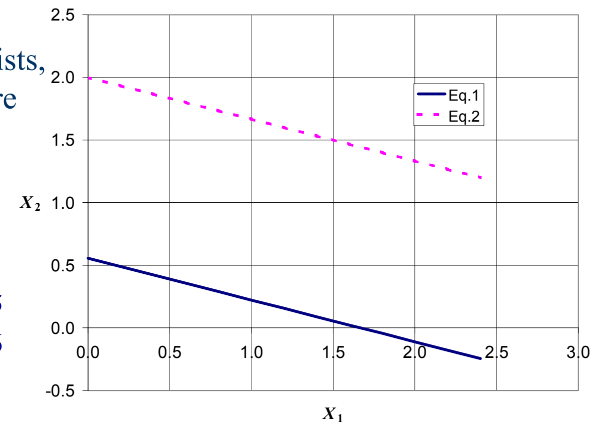
- System without a Solution

No solution exists, the two lines are parallel



$$\text{Eq. 1}: 3X_1 + 9X_2 = 5$$
$$\text{Eq. 2}: X_1 + 3X_2 = 6$$

Simultaneous Linear Equations

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- System with an Infinite Number of Solutions
  - Consider the following two system of simultaneous equation:

$$2X_1 + 3X_2 = 4$$
$$4X_1 + 6X_2 = 8$$

  - This system has infinite number of solutions

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{6(4) - 3(8)}{2(6) - 4(3)} = \frac{0}{0} \qquad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{2(8) - 4(4)}{2(6) - 4(3)} = \frac{0}{0}$$

Simultaneous Linear Equations

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■ System with an Infinite Number of Solutions

$$\text{Eq.1: } 2X_1 + 3X_2 = 4$$
$$\text{Eq. 2: } 4X_1 + 6X_2 = 8$$

$$X_2 = \frac{4 - 2X_1}{3}$$
$$X_2 = \frac{8 - 4X_1}{6}$$

X_1	Eq.1	Eq.2
0.0	1.333	1.333
0.2	1.200	1.200
0.4	1.067	1.067
0.6	0.933	0.933
0.8	0.800	0.800
1.0	0.667	0.667
1.2	0.533	0.533
1.4	0.400	0.400
1.6	0.267	0.267
1.8	0.133	0.133
2.0	0.000	0.000
2.2	-0.133	-0.133
2.4	-0.267	-0.267

Simultaneous Linear Equations

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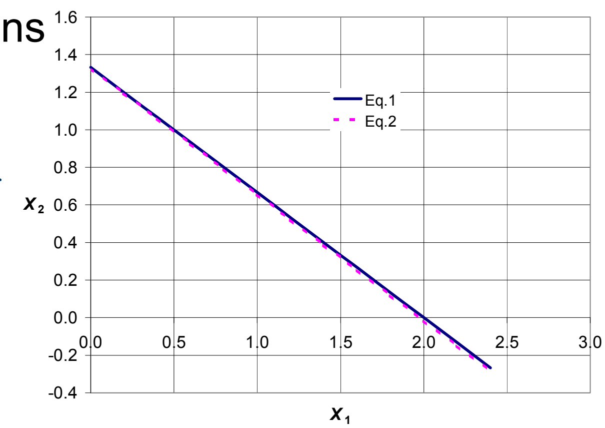
■ System with an Infinite Number of Solutions

The intersection of these two lines is defined by the entire line; therefore, there are infinite number of solutions.



$$\text{Eq.1: } 2X_1 + 3X_2 = 4$$
$$\text{Eq. 2: } 4X_1 + 6X_2 = 8$$

Simultaneous Linear Equations

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■ ILL-conditioned System

– Consider the following two system of simultaneous equation:

$$2X_1 + 2.2X_2 = 5.7$$
$$2X_1 + 2X_2 = 5.5$$

– This system has infinite number of solutions

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{2(5.7) - 2.2(5.5)}{2(2) - 2(2.2)} = 1.75 \qquad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{2(5.5) - 2(5.7)}{2(2) - 2(2.2)} = 1$$

Simultaneous Linear Equations

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■ ILL-conditioned System

$$\text{Eq.1: } 2X_1 + 2.2X_2 = 5.7$$
$$\text{Eq.2: } 2X_1 + 2X_2 = 5.5$$

$$X_2 = \frac{5.7 - 2X_1}{2.2}$$
$$X_2 = \frac{5.5 - 2X_1}{2}$$

X_1	Eq.1	Eq.2
0.0	2.591	2.750
0.2	2.409	2.550
0.4	2.227	2.350
0.6	2.045	2.150
0.8	1.864	1.950
1.0	1.682	1.750
1.2	1.500	1.550
1.4	1.318	1.350
1.6	1.136	1.150
1.8	0.955	0.950
2.0	0.773	0.750
2.2	0.591	0.550
2.4	0.409	0.350

Simultaneous Linear Equations

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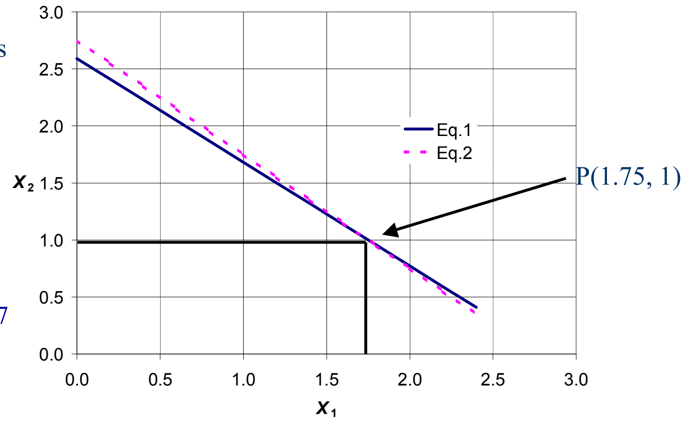
■ ILL-conditioned System

While the solution exists for this system, the two lines has almost the same slope, which can produce computational difficulties in numerical methods.

$$\text{Eq.1}: \; 2X_1 + 2.2X_2 = 5.7$$
$$\text{Eq.2}: \; 2X_1 + 2X_2 = 5.5$$



Simultaneous Linear Equations

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■ Numerical Procedures
- Gaussian Elimination
- Gauss-Jordan Elimination
- LU Decomposition
- Iterative Equation-Solving Methods
  - Jacobi Iteration
  - Gauss-Seidel Iteration
- Use of Determinants

Simultaneous Linear Equations

- Matrix Representation of the System of Equations
  - A solution of two or three simultaneous equations does not present a problem of notation.
  - However, the solution of a larger set of *n* simultaneous equations can present a problem of notations, and they can be difficult to manage.

Simultaneous Linear Equations

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- Matrix Representation of the System of Equations
  - Therefore, for large systems of equations, the set of equations can be simplified by presenting it in matrix form.
  - A set of simultaneous equations can be, for example, presented in a matrix form as follows:

$$[A][X]=[C]$$

Simultaneous Linear Equations

- Matrix Representation of the System of Equations

$$[A][X]=[C] \qquad (1)$$

[***A***] = coefficient matrix

[***X***] = column vector of unknowns

[***C***] = column vector of constants

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- Matrix Representation of the System of Equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \qquad (2)$$

Simultaneous Linear Equations

- Matrix Representation of the System of Equations
  - Equation 2 can be expressed in a more compact and convenient form by dropping the unknown X_i terms and incorporating the constant C_i terms as an additional column in the coefficient matrix as shown in the next slide.

Simultaneous Linear Equations

- Compact Matrix Representation of the System of Equations

$$[A][X]=[C]$$

$$\downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & C_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & C_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & C_n \end{bmatrix} \qquad (3)$$

Simultaneous Linear Equations

- Example: (Matrix Form)

  Put the following set of simultaneous equations in a matrix form:

$$
\begin{aligned}
2X_1 - 4X_2 + 6X_3 &= 5 \\
X_1 + 3X_2 - 7X_3 &= 2 \\
7X_1 + 5X_2 + 9X_3 &= 4
\end{aligned}
$$

Simultaneous Linear Equations

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- Example (cont'd): (Matrix Form)
  - In general matrix form, the result is

$$
\begin{aligned}
2X_1 - 4X_2 + 6X_3 &= 5 \\
X_1 + 3X_2 - 7X_3 &= 2 \\
7X_1 + 5X_2 + 9X_3 &= 4
\end{aligned}
\quad \Longrightarrow \quad
\begin{bmatrix} 2 & -4 & 6 \\ 1 & 3 & -7 \\ 7 & 5 & 9 \end{bmatrix}
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}
=
\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}
$$

Simultaneous Linear Equations

- Example (cont'd): (Matrix Form)
  - In a more compact matrix form, the result is

$$\begin{aligned} 2X_1 - 4X_2 + 6X_3 &= 5 \\ X_1 + 3X_2 - 7X_3 &= 2 \\ 7X_1 + 5X_2 + 9X_3 &= 4 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 2 & -4 & 6 & 5 \\ 1 & 3 & -7 & 2 \\ 7 & 5 & 9 & 4 \end{bmatrix}$$

Gaussian Elimination

- Gaussian elimination is one of the most popular and efficient methods of solving an $n \times n$ system of equation.
- The method is relatively simple and straightforward.
- It consists of a series of operations to transform the original set to a new system.

Gaussian Elimination

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- The new system consists of n simultaneous equations in n unknowns having a triangular form from which each unknown is determined by back-substitution.
- Details of the procedure are provided later after the concept of permissible operations has been introduced.

Gaussian Elimination

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- Permissible Operations
  - Permissible operations are mathematical operations on a set of simultaneous equations.
  - These operations do not alter the solution.
  - Three permissible operations are of interest herein; they are described in the context of simple, two-equation system.

Gaussian Elimination

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■ Permissible Operations

First:

- The solution to a set of simultaneous equations does not change if the order in which the equations are written is changed.

Second:

- Any one of the equation can be multiplied or divided by a nonzero constant without changing the solution.

Gaussian Elimination

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■ Permissible Operations

Third:

- It is permissible to add two equations together and use the resulting equation to replace either of the two original equations.

Gaussian Elimination

- Example (First Operation):
  - If the original set is written as

$$\begin{aligned} 2X_1 + 3X_2 &= 1 \\ -4X_1 + X_2 &= 5 \end{aligned} \quad \text{(original set)}$$

  - Then the following set will not change the solution to the original set:

$$\begin{aligned} -4X_1 + X_2 &= 5 \\ 2X_1 + 3X_2 &= 1 \end{aligned}$$

Gaussian Elimination

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- Example (Second Operation):
  - If the original set is given as

$$\begin{aligned} 2X_1 + 3X_2 &= 1 \\ -4X_1 + X_2 &= 5 \end{aligned} \quad \text{(original set)}$$

  - And if the first equation in the original set is multiplied by 2, then the new set is:

$$\begin{aligned} 4X_1 + 6X_2 &= 2 \\ -4X_1 + X_2 &= 5 \end{aligned}$$

Gaussian Elimination

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■ Example (Second Operation): (cont'd)

– The solution to the original set is given by

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{1(1)-3(5)}{2(1)-(-4)(3)} = -1 \qquad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{2(5)-(-4)(1)}{2(1)-(-4)(3)} = 1$$

– The solution to the new set, which is identical to the first, is given by

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} = \frac{1(2)-6(5)}{4(1)-(-4)(6)} = -1 \qquad X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} = \frac{4(5)-(-4)(2)}{4(1)-(-4)(6)} = 1$$

Gaussian Elimination

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■ Example (Third Operation):

– If the original set is given as

$$\begin{aligned} 2X_1 + 3X_2 &= 1 \\ -4X_1 + X_2 &= 5 \end{aligned} \qquad \text{(original set)}$$

– And if the first equation is added to the second, then a new equation is produced as

$$-2X_1 + 4X_2 = 6$$

Gaussian Elimination

- Example (Third Operation): (cont'd)
  - The solution to the following two sets of equations will be the same as the solution to the original set:

$2X_1 + 3X_2 = 1$ $-2X_1 + 4X_2 = 6$	$-4X_1 + X_2 = 5$ $-2X_1 + 4X_2 = 6$
Solution : $X_1 = -1,\ X_2 = 1$	Solution : $X_1 = -1,\ X_2 = 1$