CHAPTER 5a. SIMULTANEOUS LINEAR EQUATIONS

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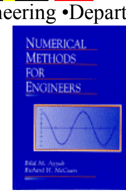

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Introduction

- Systems of simultaneous equations can be found in many engineering applications and problems.
- Systems that consist of small number of equations can be solved analytically using standard methods from algebra.
- Systems of large number of equations require the use of numerical methods and computers.

Introduction

- Simultaneous Linear Equations and Engineering
  - The system of simultaneous equations is probably one of the most important topics in modern engineering computations.
  - This is not an exaggeration if one considers that recent technological advances were made possible by the ability of solving larger and larger systems of equations.

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- Simultaneous Linear Equations and Engineering

Small Systems	Larger System
$X_1 + 3X_2 + 2X_3 = 15$ $2X_1 + 4X_2 + 3X_3 = 22$ $3X_1 + 4X_2 + 7X_3 = 39$	$X_1 + 2X_2 - 4X_3 + \cdots + 10X_{20} = 20.1$ $-2X_1 + X_2 - 3X_3 + \cdots + 2X_{20} = 2$ $2X_1 - 7X_2 - 4X_3 + \cdots - 8X_{20} = -6.5$ $\vdots$ $8X_1 - 2X_2 + 8X_3 + \cdots + 3X_{20} = -11$
$3X - 0.1Y - 0.2Z = 7.85$ $0.1X + 7Y - 0.3Z = -19.2$ $0.3X - 0.2Y + 10Z = 71.7$	

Introduction

- Simultaneous Linear Equations and Engineering
  - Technological advances in engineering that involve large number of simultaneous equations include:
    - The finite element method
    - The finite difference method
    - The analysis of structural, mechanical, and electrical systems.

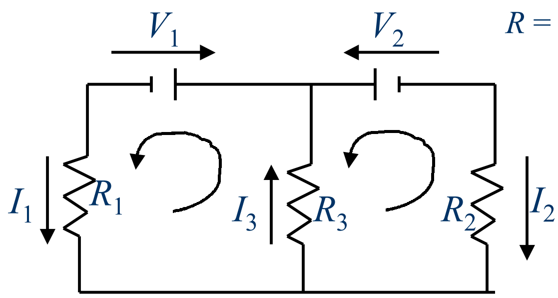
Introduction

- Simultaneous Linear Equations and Engineering
  - In the past, the numerical methods (such as the finite element and finite difference methods) that were used to solve systems of simultaneous equations were not attractive owing to the tremendous amount of calculations involved.
  - However, computers have changed that and altered our approach to engineering problem solving.

Introduction

- Engineering Examples
  - Electrical Circuit

I = current
V = voltage
R = resistance



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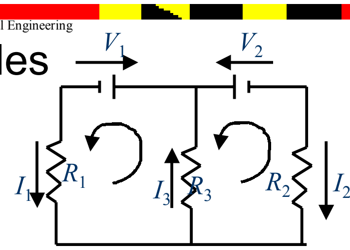
- Engineering Examples
  - Electrical Circuit



- Kirchhoff's first law states that the algebraic sum of current flowing into a junction of a circuit must equal zero
- Kirchhoff's second law states that the algebraic sum of the electromotive forces around a closed circuit must equal the sum of voltage drops around the circuit.

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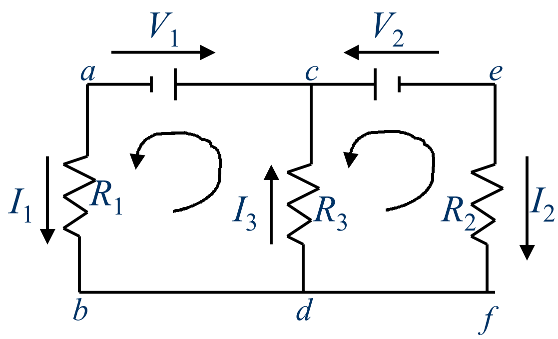
■ Engineering Examples

– Electrical Circuit



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■ Engineering Examples

– Electrical Circuit

- Applying Kirchhoff's first law at junction *c*, yields the following linear equation:

$$I_1 + I_2 - I_3 = 0$$

- Applying Kirchhoff's second law to loop *acdb* yields the following linear equation:

$$V_1 = R_1 I_1 + R_3 I_3$$

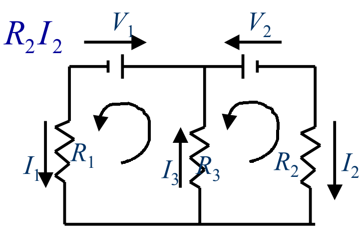
Introduction

■ Engineering Examples

– Electrical Circuit

• Applying Kirchhoff's second law to loop *aefb* yields the following linear equation:

$$V_1 - V_2 = R_1 I_1 - R_2 I_2$$



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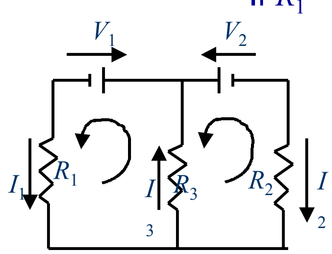
■ Engineering Examples

– Electrical Circuit

• If $R_1 = 2$, $R_2 = 4$, $R_3 = 5$, $V_1 = 6$, and $V_2 = 2$, then



$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 2I_1 \qquad + 5I_3 &= 6 \\ 2I_1 - 4I_2 \qquad &= 4 \end{aligned}$$

The solution to these three equations produces the current flows in the network.

Introduction

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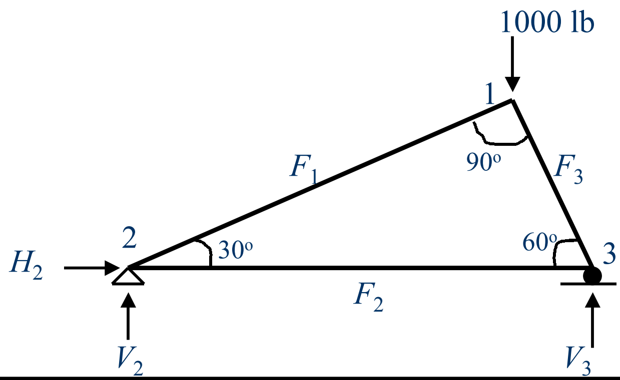
■ Engineering Examples

– Analysis of Statically Determinant Truss



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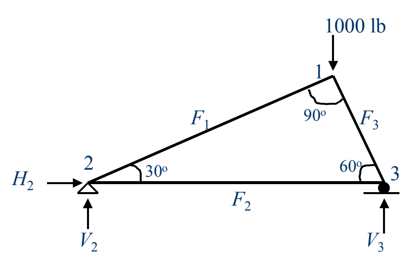
■ Engineering Examples

– Analysis of Statically Determinant Truss

$$\sum F_H = H_2 = 0$$
$$\sum F_V = V_1 + V_2 - 1000 = 0$$



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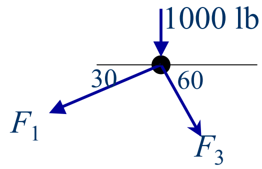
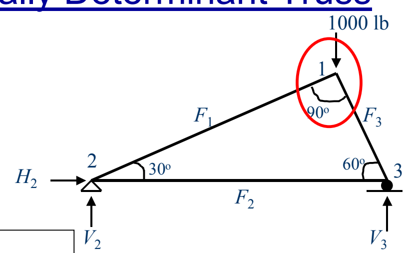
- Engineering Examples
  - Analysis of Statically Determinant Truss





$$\sum F_H = -F_1 \cos 30 + F_3 \cos 60 = 0$$
$$\sum F_V = -1000 - F_1 \sin 30 - F_3 \sin 60 = 0$$
(1)

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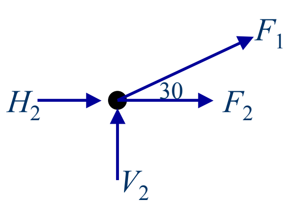
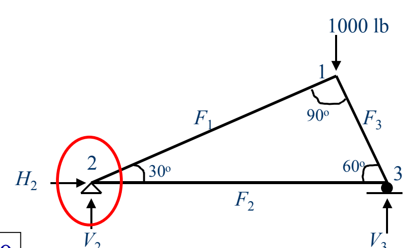
- Engineering Examples





0

$$\sum F_H = H_2 + F_1 \cos 30 + F_2 = 0$$
$$\sum F_V = V_2 + F_1 \sin 30 = 0$$
(2)

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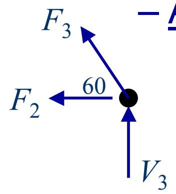
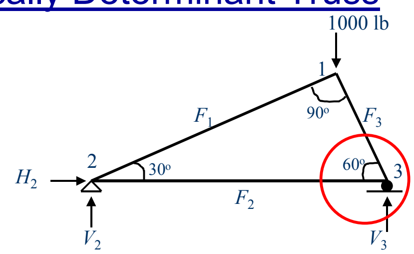
■ Engineering Examples

– Analysis of Statically Determinant Truss





$$\sum F_H = -F_2 - F_3 \cos 60 = 0$$
$$\sum F_V = V_3 + F_3 \sin 60 = 0 \quad (3)$$

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■ Engineering Examples

– Analysis of Statically Determinant Truss

- Combining the three systems of equations (systems 1, 2, and 3), we obtain one system of simultaneous equation as shown in the next viewgraph:

$$\sum F_H = -F_1 \cos 30 + F_3 \cos 60 = 0$$
$$\sum F_V = -1000 - F_1 \sin 30 - F_3 \sin 60 = 0$$

$$\sum F_H = -F_2 - F_3 \cos 60 = 0$$
$$\sum F_V = V_3 + F_3 \sin 60 = 0$$

$$\sum F_H = H_2 + F_1 \cos 30 + F_2 = 0$$
$$\sum F_V = V_2 + F_1 \sin 30 = 0$$

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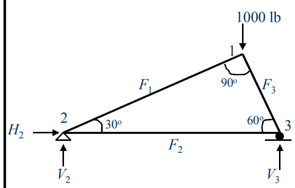
- Engineering Examples
  - Analysis of Statically Determinant Truss



$$
\begin{aligned}
-0.866F_1 + 0.5F_3 &= 0 \\
-0.5F_1 - 0.866F_3 &= 1000 \\
0.866F_1 + F_2 + H_2 &= 0 \\
0.5F_1 + V_2 &= 0 \\
-F_2 - 0.5F_3 &= 0 \\
0.866F_3 + V_3 &= 0
\end{aligned}
$$

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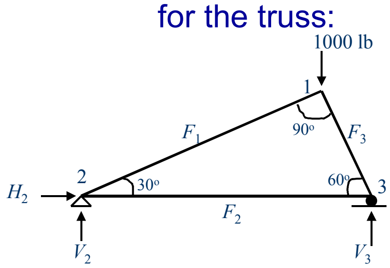
- Engineering Examples
  - Analysis of Statically Determinant Truss
    - The solution to the previous system of equations provides the following force values for the truss:



$$
\begin{aligned}
F_1 &= -500 \text{ lb} \\
F_2 &= 433 \text{ lb} \\
F_3 &= -866 \text{ lb} \\
H_2 &= 0 \text{ (as expected)} \\
V_2 &= 250 \text{ lb} \\
V_3 &= 750 \text{ lb}
\end{aligned}
$$

Introduction

- Engineering Examples
  - Engineering Dynamics Example:
    - Suppose that a team of three parachutists is connected by a weightless cord while free-falling at a velocity of 5 m/s as shown in the figure. Compute the tension in each section of the cord and the acceleration of the team, given the masses of each parachutist and the dreg coefficients as provided in the table.

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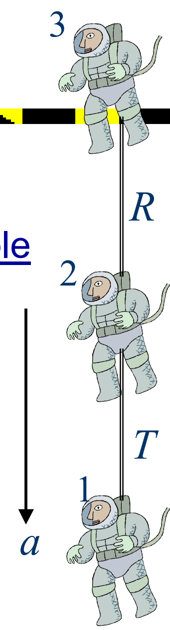
- Engineering Examples
  - Engineering Dynamics Example

Parachutist	Mass (kg)	Drag Coefficient (kg/s)
1	80	11
2	70	15
3	50	18



Introduction

■ Engineering Examples

– Engineering Dynamics Example

Free-body diagrams are needed for each of the parachutists as shown in the next viewgraph.

Summing forces in the vertical direction and using Newton's second law of motion, gives:

$$\begin{aligned} m_1 g - T - c_1 v \qquad &= m_1 a \\ m_2 g + T - c_2 v - R &= m_2 a \qquad (4) \\ m_3 g \qquad - c_3 v + R &= m_3 a \end{aligned}$$

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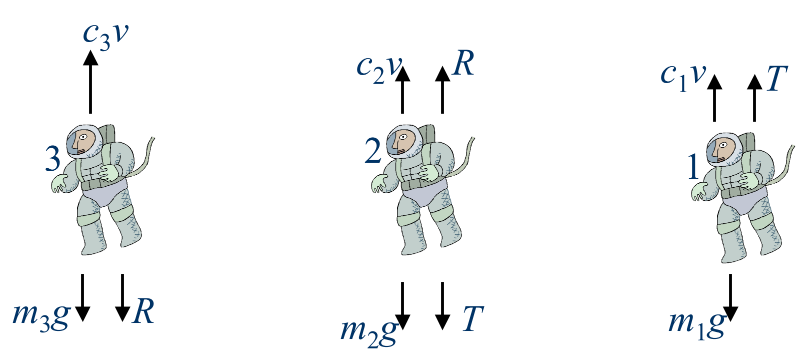
■ Engineering Examples

– Engineering Dynamics Example



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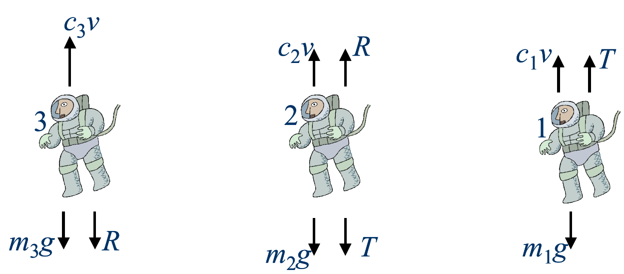
- Engineering Examples
  - Engineering Dynamics Example



$m_3 g + R - c_1 v = m_3 a$ | $m_2 g + T - c_2 v - R = m_2 a$ | $m_1 g - c_1 v - T = m_1 a$

Introduction

- Engineering Examples
  - Engineering Dynamics Example

    Substituting the values for parachutists masses and drag coefficients, the system of equations provided by Eq. 4, gives

$$
\begin{aligned}
80(9.8) - T - 11(5) \quad &= 80a \\
70(9.8) + T - 15(5) - R &= 70a \\
50(9.8) \quad - 18(5) + R &= 50a
\end{aligned}
$$

Introduction

- Engineering Examples
  - Engineering Dynamics Example

    After rearranging and simplifying,

$$\begin{aligned} 80a + T \quad &= 729 \\ 70a - T + R &= 611 \\ 50a \quad - R &= 400 \end{aligned}$$

The solution to the above system of equations gives the following values:

$$a = 8.7\ \text{m/s}^2, \qquad T = 33\ \text{N}, \quad \text{and} \ \ R = 35\ \text{N}$$

General Form for a System of Equations

- Definition

  A *linear equation* is one in which a variable only appears to the first power in every term of a given equation.

  Thus, a system of m linear equations in n unknowns X_j, j = 1, 2, …,n, can be represented as

$$\sum_{j=1}^{n} a_{ij} X_j = C_i, \qquad i = 1, 2, \cdots, m$$

General Form for a System of Equations

■ Expanded Form

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= C_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= C_2 \\ &\vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= C_m \end{aligned} \quad (5a)$$

a_{ij} = known coefficients of the equations
X_j = unknown variables
C_i = known constants

General Form for a System of Equations

■ Expanded Form ($m = n$)

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= C_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= C_2 \\ &\vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n &= C_n \end{aligned} \quad (5b)$$

a_{ij} = known coefficients of the equations
X_j = unknown variables
C_i = known constants

General Form for a System of Equations

- Classification of Systems of Equations
  1. A set of equations in which the number of unknowns is equal to the number of equations ($n = m$)
  2. A set of equations in which the number of unknowns is less than the number of equations ($n < m$)
  3. A set of equations in which the number of unknowns is greater than the number of equations ($n > m$)

Solution of a System of Two Equations

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- The solution of two equation gives insight to understand the classification of systems of equations based on graphical interpretation in two-dimensional space.
  – If n equals 2, then Eq. 5 reduces to

$$a_{11}X_1 + a_{12}X_2 = C_1 \qquad (6a)$$

$$a_{21}X_1 + a_{22}X_2 = C_2 \qquad (6b)$$

Solution of a System of Two Equations

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- A solution for simple system can be obtained by substitution.
- Solving Eq. 6a for X_1 gives

$$X_1 = \frac{C_1 - a_{12}X_2}{a_{11}} \qquad (7a)$$

- The expression for X_1 in the above equation can be substituted into Eq. 6b:

$$a_{21}\frac{C_1 - a_{12}X_2}{a_{11}} + a_{22}X_2 = C_2 \qquad (7b)$$

Solution of a System of Two Equations

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- Equation 7b is a single equation with one unknown, X_2.
- This equation can be solved for X_2 to give

$$X_2 = \frac{a_{11}C_2 - a_{21}C_1}{a_{11}a_{22} - a_{21}a_{12}} \qquad (8a)$$

- Eq. 8a can be substituted back into Eq. 7a to give

$$X_1 = \frac{a_{22}C_1 - a_{12}C_2}{a_{11}a_{22} - a_{21}a_{12}} \qquad (8b)$$

Solution of a System of Two Equations

– It seems that the solution procedure for this set of equations is simple to apply.
– One should imagine the effort that would be required to solve 15 or 20 simultaneous equations using the substitution procedure.
– Many complex engineering problems involve hundreds or even thousands of simultaneous equations.
– Hence, the need for alternative solution procedures is justified.

Types of Numerical Procedures

■ Because of the wide-spread use of computers nowadays, numerical solution methods are widely used. There are three general types:

1. Elimination methods,
2. Iteration methods, and
3. Method of determinants.

Classification of Systems of Equations Based on Graphical Interpretation

- Systems of equations can be classified based on their solutions to the following types:
  1. Systems that have solutions,
  2. Systems without solution, and
  3. Systems with an infinite number of solutions.