# CHAPTER 4e.
# ROOTS OF EQUATIONS

NUMERICAL METHODS FOR ENGINEERS

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by

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## Polynomial Reduction

- Example 2

  Newton-Raphson iteration resulted in a root of $x_1$ equals to 1.1211 for the following polynomial: $x^4 - 15x^2 - 6x + 24$

  Find a reduced polynomial.

# Polynomial Reduction

- Example 2 (cont'd)

$$
\begin{array}{r|l}
 & x^3 + 1.1211x^2 - 13.7431x - 21.4074 \\
\hline
x - 1.1211 & x^4 \qquad - 15x^2 \quad - 6x + 24 \\
 & x^4 - 1.1211x^3 \\
\cline{2-2}
 & \quad 1.1211x^3 - 15.0000x^2 - 6x + 24 \\
 & \quad 1.1211x^3 - \quad 1.2569x^2 \\
\cline{2-2}
 & \qquad - 13.7431x^2 \quad - 6.0000x + 24 \\
 & \qquad - 13.7431x^2 + 15.4074x \\
\cline{2-2}
 & \qquad\qquad - 21.4074x + 24 \\
 & \qquad\qquad - 21.4074x + 23.9998 \\
\cline{2-2}
 & \qquad\qquad\qquad 0.0002 = \text{error}
\end{array}
$$

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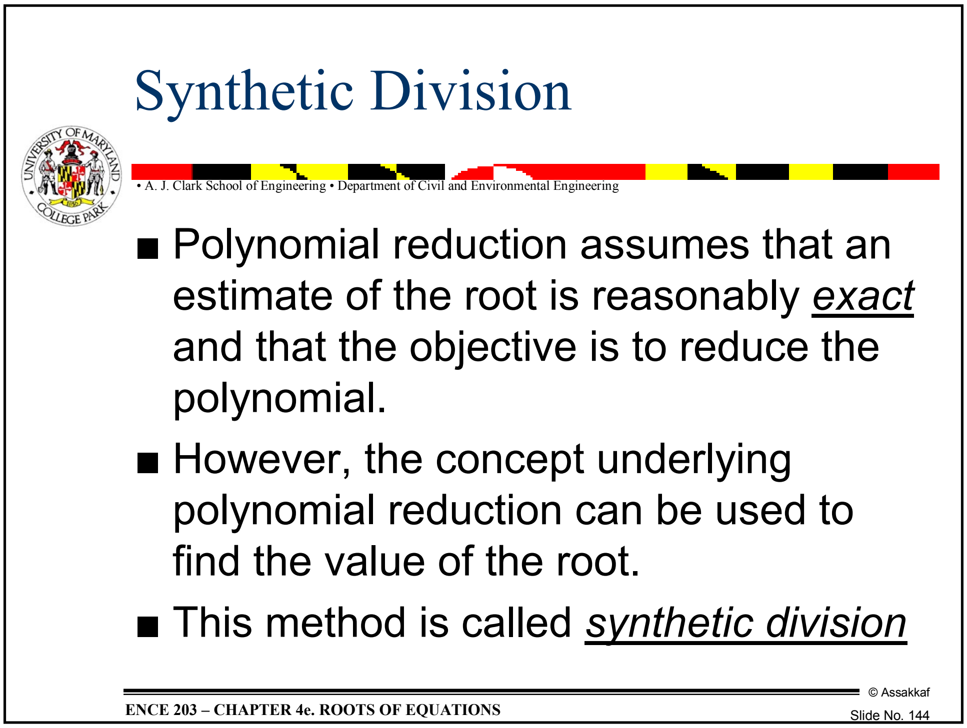
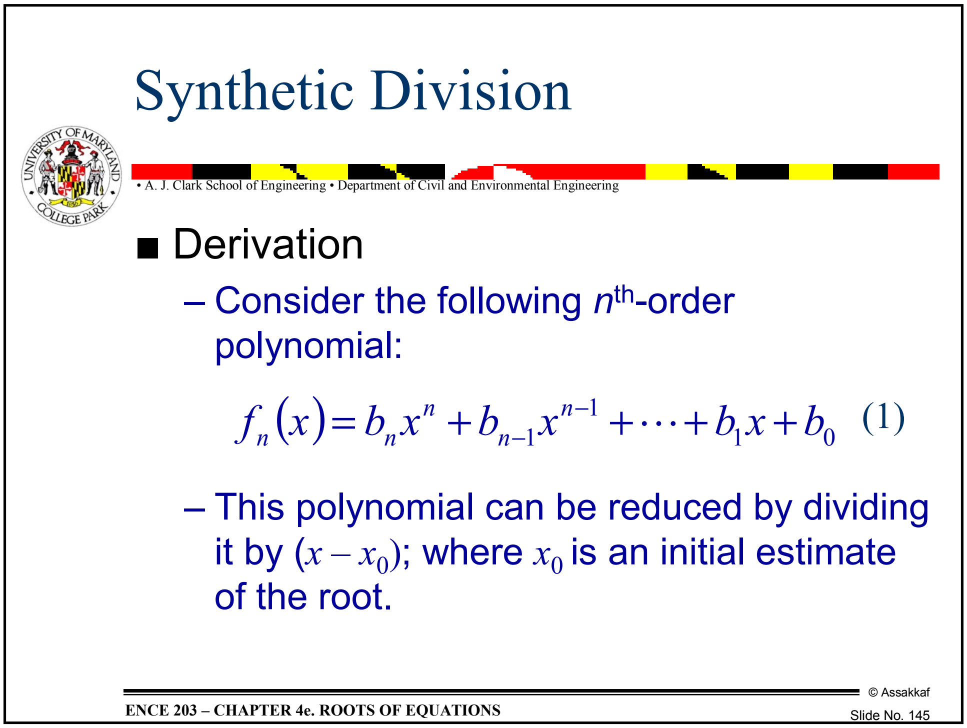
# Polynomial Reduction

- Example 2 (cont'd)
  - The reduced polynomial $x^3 + 1.1211x^2 - 13.7431x - 21.4074$ can be used to find additional roots for the original polynomial $x^4 - 15x^2 - 6x + 24$ .
  - Any other method then can be used to find a root of the reduced polynomial, and the polynomial can be reduced again using polynomial reduction until all of the roots are found.



## Synthetic Division

- Polynomial reduction assumes that an estimate of the root is reasonably *exact* and that the objective is to reduce the polynomial.
- However, the concept underlying polynomial reduction can be used to find the value of the root.
- This method is called *synthetic division*



## Synthetic Division

- Derivation
  - Consider the following $n^{\text{th}}$-order polynomial:

$$f_n(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \quad (1)$$

  - This polynomial can be reduced by dividing it by $(x - x_0)$; where $x_0$ is an initial estimate of the root.

# Synthetic Division

- Derivation (cont'd)
  - The reduced polynomial $h_{n\text{-}1}(x)$ is of order $n$ –1, and therefore

$$\frac{f_n(x)}{x-x_0}=h_{n-1}(x)+\frac{R_0}{x-x_0} \qquad (2)$$

where $R_0$ is the remainder.

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# Synthetic Division

- Derivation (cont'd)
  - Equation 2 can be written as

$$f_n(x)=(x-x_0)h_{n-1}(x)+R_0 \qquad (3)$$

  - The reduced polynomial $h_{n\text{-}1}(x)$ is given by

$$h_{n-1}(x)=c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\cdots+c_1x+c_0 \qquad (4)$$

# Synthetic Division

- Derivation (cont'd)
  - If $h_{n\text{-}1}(x)$ is also reduced using the estimate of the root, the following ($n$-2)th-order polynomial designated $g_{n\text{-}2}(x)$ can be found as

$$\frac{h_{n-1}(x)}{x-x_0} = g_{n-2}(x) + \frac{R_1}{x-x_0} \qquad (5)$$

where $R_1$ is the remainder.

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# Synthetic Division

- Derivation (cont'd)
  - The reduced polynomial $g_{n\text{-}2}(x)$ can be expressed as

$$g_{n-2}(x) = d_{n-2}x^{x-2} + d_{n-3}x^{x-3} + \cdots + d_1 x + d_0 \qquad (6)$$

  - Recall Eq. 3

$$f_n(x) = (x-x_0)h_{n-1}(x) + R_0$$

# Synthetic Division

- Derivation (cont’d)
  - Eq. 3 can rewritten as

$$h_{n-1}(x)=\frac{f_n(x)}{(x-x_0)}-\frac{R_0}{(x-x_0)} \qquad (7)$$

  - Substituting $h_{n-1}(x)$ of Equation 7 into Equation 5, yields the following results:

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# Synthetic Division

- Derivation (cont’d)

$$\frac{h_{n-1}(x)}{x-x_0}=g_{n-2}(x)+\frac{R_1}{x-x_0}$$

$$\frac{\dfrac{f_n(x)}{(x-x_0)}-\dfrac{R_0}{(x-x_0)}}{x-x_0}=g_{n-2}(x)+\frac{R_1}{x-x_0}$$

$$\frac{f_n(x)}{(x-x_0)^2}-\frac{R_0}{(x-x_0)^2}=g_{n-2}(x)+\frac{R_1}{x-x_0} \qquad (8)$$

# Synthetic Division

- Derivation (cont'd)

$$\boxed{\frac{f_n(x)}{(x-x_0)^2}-\frac{R_0}{(x-x_0)^2}=g_{n-2}(x)+\frac{R_1}{x-x_0}}$$

  - Multiplying both sides of Eq. 8 by $(x-x_0)^2$ and rearranging, gives

$$f_n(x)=(x-x_0)^2 g_{n-2}(x)+(x-x_0)R_1+R_0 \qquad (9)$$

  - The derivative of Eq. 9 with respect to $x$ is

$$f_n'(x)=2(x-x_0)g_{n-2}(x)+(x-x_0)^2 g_{n-2}'(x)+R_1 \qquad (10)$$

where $g_n'(x)$ = derivative with respect to $x$.

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# Synthetic Division

- Derivation (cont'd)
  - Recall Equations 9 and 10, respectively:

$$f_n(x)=(x-x_0)^2 g_{n-2}(x)+(x-x_0)R_1+R_0$$

$$f_n'(x)=2(x-x_0)g_{n-2}(x)+(x-x_0)^2 g_{n-2}'(x)+R_1$$

  - At $x=x_0$, Eqs. 9 and 10 reduce to

$$f_n(x)=R_0 \qquad (11)$$

$$f_n'(x)=R_1 \qquad (12)$$

# Synthetic Division

- Derivation (cont'd)
  - Thus, based on Newton-Raphson method, the following expression can be obtained:

$$x_1 = x_0 - \frac{f_n(x)}{f_n'(x)} = x_0 - \frac{R_0(x_i)}{R_1(x_i)} \qquad (13)$$

where $R_0(x_i)$ is the remainder term based on an initial polynomial reduction with a root of $x_i$ and $R_1(x_i)$ is the remainder term based on second polynomial reduction using again $x_i$.

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# Synthetic Division

- Synthetic Division Iterative Equation
  - Eq. 13 indicates that, for an initial estimate of the root $x_0$ an improved estimate can be obtained with Eq. 13 after computing $R_0$ and $R_1$ by polynomial reduction. The iterative root estimation is given by

$$x_{i+1} = x_i - \frac{R_0(x_i)}{R_1(x_i)} \qquad (14)$$

# Synthetic Division

- Programming Considerations
  - The solution procedure for synthetic division is easily programmed using the general form of Eqs. 1 to 13.
  - By dividing $f_n(x)$, Eq. 1, by $(x - x_0)$, we get

$$\begin{array}{r|l} & c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + \cdots + c_0 \\ \cline{2-2} x - x_0 & b_n x^n \qquad + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0 \end{array}$$

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# Synthetic Division

- Programming Considerations
  - It can be shown that the coefficients of the reduced polynomial $c_i$ are related to the $b_i$ values by

$$\boxed{\begin{array}{l} c_n = b_n \\ c_j = b_j + x_i c_{j+1} \qquad \text{for } j = (n-1), (n-2), \cdots, 1 \end{array}} \qquad (15)$$

Where $x_i$ is the estimate of the root in the $i^{\text{th}}$ iteration.

# Synthetic Division

- Programming Considerations
  - Similarly, the coefficients of Eq. 6 are related to those of Eq. 4 by

$$\boxed{\begin{aligned} &d_n = c_n \\ &d_j = c_j + x_i d_{j+1} \qquad \text{for } j = (n-1),(n-2),\cdots,1 \end{aligned}} \qquad (16)$$

# Synthetic Division

- Programming Considerations
  - The remainders $R_0$ and $R_1$ in the $i^{\text{th}}$ iteration are given by

$$\boxed{\begin{aligned} R_0 &= b_0 + x_i c_1 \\ R_1 &= c_1 + x_i d_2 \end{aligned}} \qquad (17)$$

## Synthetic Division

- Procedure for Synthetic Division
  1. Input $n$, $x_0$, $b_j$ for $j = 0, 1, \ldots, n$.
  2. Compute the $n$ values of $c_j$ using Eq. 15.
  3. Compute $R_0$.
  4. Compute the $(n - 1)$ values of $d_j$ using Eq. 16.
  5. Compute $R_1$.
  6. Use Eq. 14 to compute a revised estimate of $x_{i+1}$ of the root

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## Synthetic Division

- Procedure for Synthetic Division
  7. Check for convergence as follows:
     a. If $|x_{i+1} - x_i| \leq$ tolerance, discontinue the iteration and use $x_{i+1}$ as the best estimate of the root; or
     b. If $|x_{i+1} - x_i| >$ tolerance, set $x_i = x_{i+1}$ and go to step 2 and continue the iteration process.

# Synthetic Division

- Example 1: Synthetic Division

Using synthetic division, find the three roots of the following polynomial:

$$x^3 - x^2 - 10x - 8 = 0$$

Use an initial estimate of $x_0 = 6$ for the first root.

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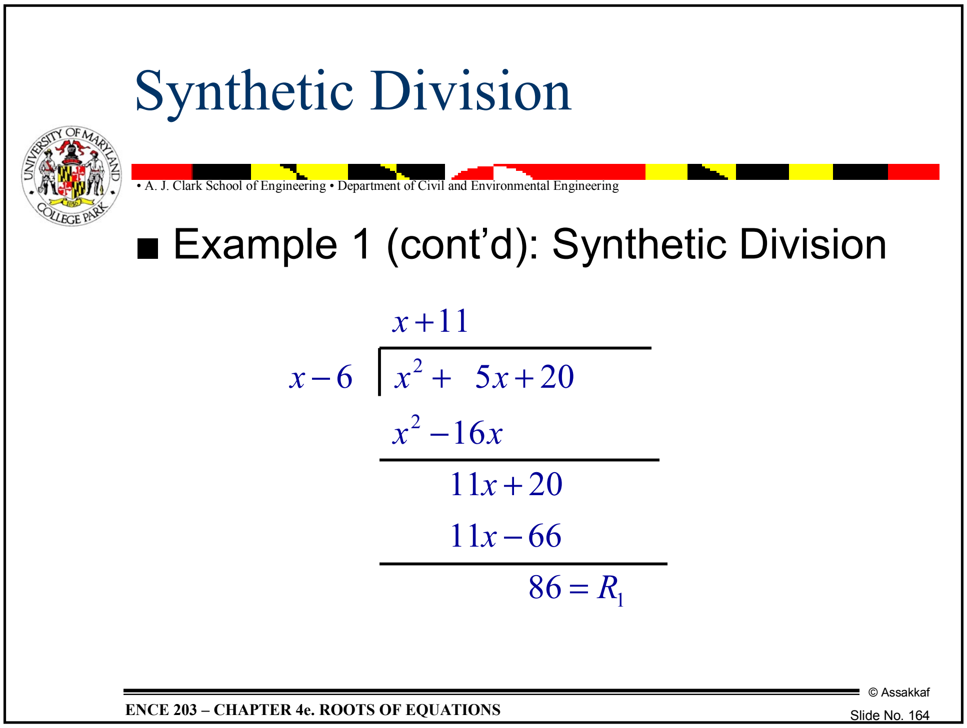
# Synthetic Division

- Example 1 (cont'd): Synthetic Division

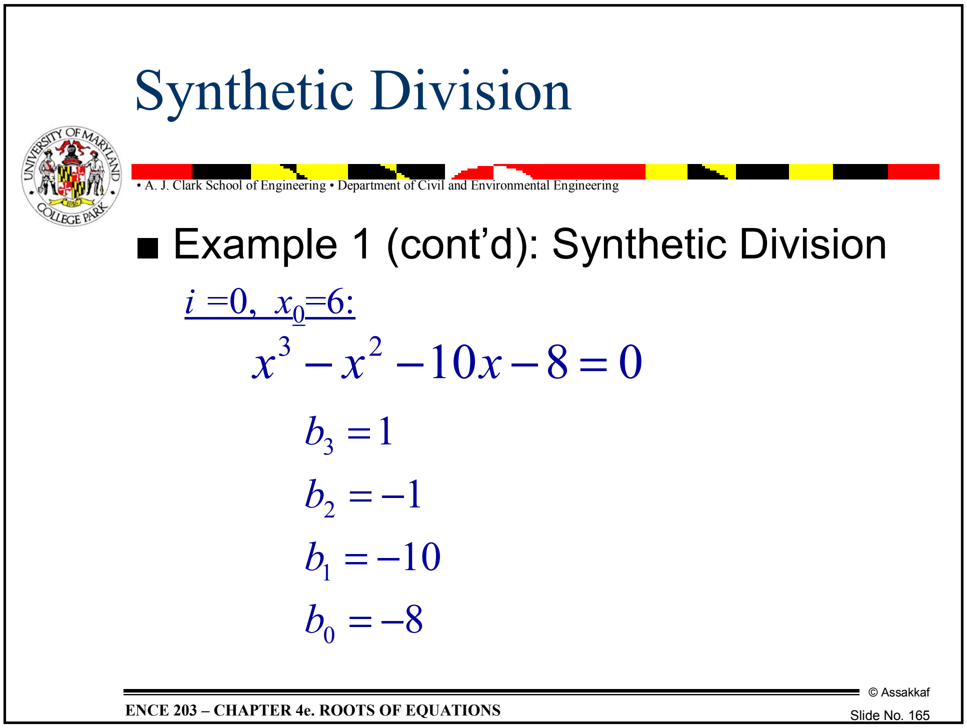
$$\begin{array}{r|l}
 & x^2 + 5x + 20 \\
\hline
x-6 & x^3 - \ \ x^2 - 10x - 8 \\
 & x^3 - 6x^2 \\
\hline
 & \quad 5x^2 - 10x \\
 & \quad 5x^2 - 30x \\
\hline
 & \qquad\quad 20x - 8 \\
 & \qquad\quad\ 20 - 120 \\
\hline
 & \qquad\qquad\quad 112 = R_0
\end{array}$$



# Synthetic Division

- Example 1 (cont’d): Synthetic Division

$$\begin{array}{r|l} & x+11 \\ \cline{2-2} x-6 & x^2+\ \ 5x+20 \\ & x^2-16x \\ \cline{2-2} & \quad 11x+20 \\ & \quad 11x-66 \\ \cline{2-2} & \qquad\quad 86=R_1 \end{array}$$



# Synthetic Division

- Example 1 (cont’d): Synthetic Division

$\underline{i=0,\ \ x_0=6:}$

$$x^3-x^2-10x-8=0$$

$$b_3=1$$

$$b_2=-1$$

$$b_1=-10$$

$$b_0=-8$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - Note that $R_0$ and $R_1$ can be found using Eqs. 15, 16, and 17 as follows:

$$\boxed{\begin{array}{l} c_n = b_n \\ c_j = b_j + x_i c_{j+1} \qquad \text{for } j = (n-1), (n-2), \cdots, 1 \end{array}}$$

$$c_3 = b_3 = 1$$

$$c_2 = b_2 + 6c_3 = -1 + 6(1) = 5$$

$$c_1 = b_1 + 6c_2 = -10 + 6(5) = 20$$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division

$$\boxed{\begin{array}{l} d_n = c_n \\ d_j = c_j + x_i d_{j+1} \qquad \text{for } j = (n-1), (n-2), \cdots, 1 \end{array}}$$

$$d_3 = c_3 = 1$$

$$d_2 = c_2 + 6d_3 = 5 + 6(1) = 11$$

$$d_1 = c_1 + 6d_2 = 20 + 6(11) = 86$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division

$$\boxed{\begin{aligned} R_0 &= b_0 + x_i c_1 \\ R_1 &= c_1 + x_i d_2 \end{aligned}}$$

$$R_0 = -8 + 6(20) = -112$$
$$R_1 = 20 + 6(11) = 86$$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - Thus, the revised estimate is

$$x_{i+1} = x_i - \frac{R_0(x_i)}{R_1(x_i)}$$

$$x_1 = x_0 - \frac{R_0(x_0)}{R_1(x_0)} = 6 - \frac{112}{86} = 4.6977$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division

$i = 1,\ x_1 = 4.6977:$

$$x^3 - x^2 - 10x - 8 = 0$$

$$b_3 = 1$$

$$b_2 = -1$$

$$b_1 = -10$$

$$b_0 = -8$$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division

New reduced Ploy.

$$\begin{array}{r|l} x - 4.6977 & x^3 - \ x^2 - 10x - 8 \\ & {*}{*}{*}{*}{*}{*}{*} \\ \hline & {*}{*}{*}{*}{*}{*}{*}{*}{*} \\ \hline & \qquad\qquad 26.6241 = R_0 \end{array}$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division

Poly 2

$x - 4.6977 \overline{\big)\text{ New reduced Poly.}}$

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$46.8091 = R_1$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - Note that $R_0$ and $R_1$ can be found using Eqs. 15, 16, and 17 as follows:

$$\boxed{\begin{array}{l} c_n = b_n \\ c_j = b_j + x_i c_{j+1} \qquad \text{for } j = (n-1), (n-2), \cdots, 1 \end{array}}$$

$$c_3 = b_3 = 1$$

$$c_2 = b_2 + 6c_3 = -1 + 4.6977(1) = 3.6977$$

$$c_1 = b_1 + 6c_2 = -10 + 4.6977(3.6977) = 7.3707$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division

$$\boxed{\begin{aligned} &d_n = c_n \\ &d_j = c_j + x_i d_{j+1} \qquad \text{for } j = (n-1), (n-2), \cdots, 1 \end{aligned}}$$

$$\begin{aligned} d_3 &= c_3 = 1 \\ d_2 &= c_2 + 4.6977 d_3 = 3.6977 + 4.6977(1) = 8.3954 \\ d_1 &= c_1 + 4.6977 d_2 = 7.3707 + 4.6977(8.3954) = 46.8098 \end{aligned}$$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division

$$\boxed{\begin{aligned} R_0 &= b_0 + x_i c_1 \\ R_1 &= c_1 + x_i d_2 \end{aligned}}$$

$$\begin{aligned} R_0 &= -8 + 4.6977(7.3707) = 26.6253 \\ R_1 &= 7.3707 + 4.6977(8.3954) = 46.8098 \end{aligned}$$

# Synthetic Division

- Example 1 (cont’d): Synthetic Division
  - Thus, the revised estimate is

$$x_{i+1} = x_i - \frac{R_0(x_i)}{R_1(x_i)}$$

$$x_2 = x_1 - \frac{R_0(x_1)}{R_1(x_1)} = 4.6977 - \frac{26.6253}{46.8098} = 4.1289$$

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# Synthetic Division

- Example 1 (cont’d): Synthetic Division
  - The result for the first root are shown in the following table.
  - After six iterations, the root to seven significant digits is 4.

# Synthetic Division

- Example 1 (cont'd): Synthetic Division

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_i$ | 6 | 4.69767442 | 4.12889444 | 4.00568744 | 4.00001182 | 4.00000000 |
| $\varepsilon_r(\%)$ | --- | 27.72277228 | 13.77559985 | 3.07580171 | 0.14188998 | 0.00029559 |
| $R_0$ | 112 | 26.62407084 | 4.05172623 | 0.17097918 | 0.00035471 | 0.00000000 |
| $R_1$ | 86 | 46.80908599 | 32.88551906 | 30.12522071 | 30.00026012 | 30.00000000 |
| $b_3$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $b_2$ | -1 | -1.00000000 | -1.00000000 | -1.00000000 | -1.00000000 | -1.00000000 |
| $b_1$ | -10 | -10.00000000 | -10.00000000 | -10.00000000 | -10.00000000 | -10.00000000 |
| $b_0$ | -8 | -8.00000000 | -8.00000000 | -8.00000000 | -8.00000000 | -8.00000000 |
| | | | | | | |
| $c_3$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $c_2$ | 5 | 3.69767442 | 3.12889444 | 3.00568744 | 3.00001182 | 3.00000000 |
| $c_1$ | 20 | 7.37047052 | 2.91887487 | 2.03984442 | 2.00008276 | 2.00000000 |
| | | | | | | |
| $d_3$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $d_2$ | 11 | 8.39534884 | 7.25778888 | 7.01137488 | 7.00002365 | 7.00000000 |
| $d_1$ | 86 | 46.80908599 | 32.88551906 | 30.12522071 | 30.00026012 | 30.00000000 |

$$x^3 - x^2 - 10x - 8 = 0$$

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - To find the second root, a reduced polynomial now can be obtained from the table.
  - The shaded area in the table contains the coefficients of this reduced polynomial, that is

$$b_2 = c_3 = 1$$
$$b_1 = c_2 = 3$$
$$b_0 = c_1 = 2$$

# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - The new polynomial is

$$x^2 + 3x + 2 = 0$$

  - Following the same procedure for synthetic division, the results are shown in the following table:
  - The second root is –1.

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# Synthetic Division

- Example 1 (cont'd): Synthetic Division

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | -0.20000000 | -0.75384615 | -0.95939730 | -0.99847524 | -0.99999768 | -1.00000000 |
| $\varepsilon_r$(%) | --- | 600.00000000 | 73.46938776 | 21.42502893 | 3.91376116 | 0.15224459 | 0.00023178 |
| $R_0$ | 6 | 1.44000000 | 0.30674556 | 0.04225128 | 0.00152709 | 0.00000232 | 0.00000000 |
| $R_1$ | 5 | 2.60000000 | 1.49230769 | 1.08120539 | 1.00304952 | 1.00000464 | 1.00000000 |
| $b_2$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $b_1$ | 3 | 3.00000000 | 3.00000000 | 3.00000000 | 3.00000000 | 3.00000000 | 3.00000000 |
| $b_0$ | 2 | 2.00000000 | 2.00000000 | 2.00000000 | 2.00000000 | 2.00000000 | 2.00000000 |
| | | | | | | | |
| $c_2$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $c_1$ | 4 | 2.80000000 | 2.24615385 | 2.04060270 | 2.00152476 | 2.00000232 | 2.00000000 |
| | | | | | | | |
| $d_2$ | 1 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 | 1.00000000 |
| $d_1$ | 5 | 2.60000000 | 1.49230769 | 1.08120539 | 1.00304952 | 1.00000464 | 1.00000000 |

$$x^2 + 3x + 2 = 0$$

## Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - To find the third root, a reduced polynomial now can be obtained from the table.
  - The shaded area in the table contains the coefficients of this reduced polynomial, that is

$$b_1 = c_2 = 1$$

$$b_0 = c_1 = 2$$

## Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - The new polynomial is

$$x + 2 = 0$$

  - The root of this polynomial can easily be found as -2

# Synthetic Division

- Example 1 (cont'd): Synthetic Division
  - Therefore, the three roots of the polynomial $x^3 - x^2 - 10x - 8$ are

$$x_1 = 4$$

$$x_2 = -1$$

$$x_3 = -2$$