# CHAPTER 4b.
# ROOTS OF EQUATIONS

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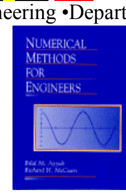


by

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**ENCE 203 - Computation Methods in Civil Engineering II**

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# Direct-Search Method

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- Trial and Error Approach
  - One method of finding the roots of a general nonlinear function is a trial-and-error approach.
  - In this approach the function $f(x)$ is evaluated at many points over some range of $x$ until $f(x)$ equals zero, that is

$$f(x) = 0$$

# Direct-Search Method



- Trial and Error Approach
  - We would have to be very lucky to guess the exact root of the function using this approach.
  - Even if the trial-and-error search is confined to a relatively small range for $x$, there are an infinite number of trial values for $x$ to be considered.





# Direct-Search Method

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- Trial and Error Approach
  - Fortunately, most problem in engineering applications do not require an exact value for the root.
  - Instead, an estimate of the root (to within some specified degree of precision) is often sufficient.

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# Direct-Search Method

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- The steps to be followed when using the direct-search method are:
  1. Specify a range or *interval* for $x$ within which the root is assumed to occur. Some knowledge of the behavior of the function is required to specify this interval. Smaller initial intervals will require fewer calculations to obtain the root to the desired accuracy.



# Direct-Search Method

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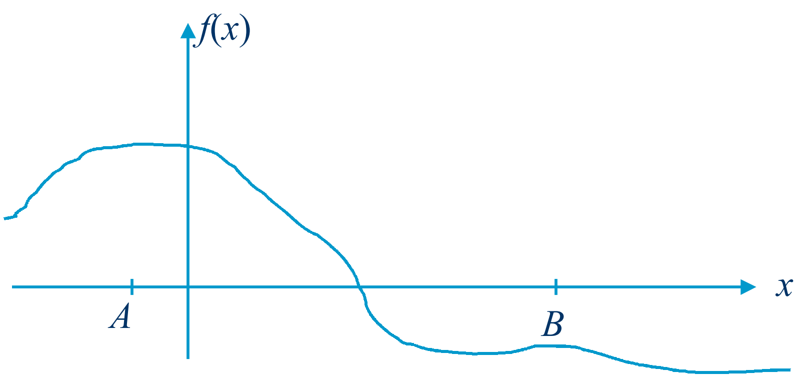
- Interval [*A*, *B*]



# Direct-Search Method



2. Divide the interval into smaller, uniformly spaced subintervals. The size of these subintervals will be dictated by the required precision for the estimate of the root. For example, if the root must be estimated to within $\pm$ 0.002, then the subintervals for the root will be 0.002 units long.





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3. Search through all subintervals until the subinterval containing the root is located . This occur when the following equality exists within the interval:

$$f(x) = 0$$

# Direct-Search Method



- Tests for Locating a Root within Subinterval
  - A simple test for determining if the root occurs within a given interval [*A*, *B*] is as follows:

    The function $f(x)$ is evaluated at the beginning and end points *A* and *B*, respectively, of the interval with the following results:



# Direct-Search Method

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- Tests for Locating a Root within Subinterval
  1. If $f(A)$ and $f(B)$ have the same sign, that is either both negative or both positive, then the root does not exist. The root will not lie within the interval.

# Direct-Search Method

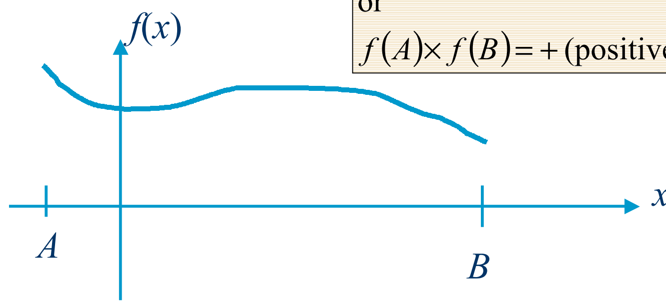


- Tests for Locating a Root within Subinterval

$f(A) > 0$ and $f(B) > 0$

or

$f(A) \times f(B) = +\text{(positive)}$






# Direct-Search Method

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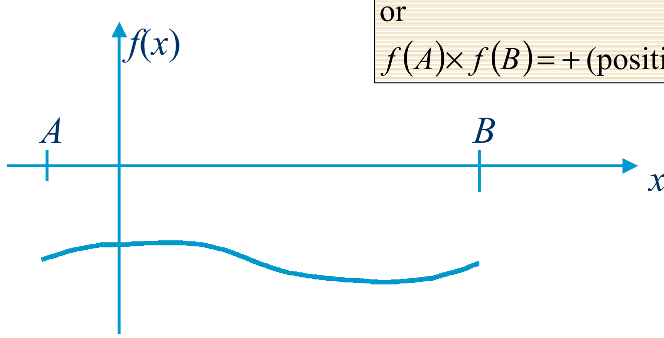
- Tests for Locating a Root within Subinterval

$f(A) < 0$ and $f(B) < 0$

or

$f(A) \times f(B) = +\text{(positive)}$



# Direct-Search Method



- Tests for Locating a Root within Subinterval

  2. If $f(A)$ and $f(B)$ have different signs, that is either $f(A)$ is positive and $f(B)$ is negative or either $f(A)$ is negative and $f(B)$ is positive , then the root does exist (or lies) within the interval.



# Direct-Search Method

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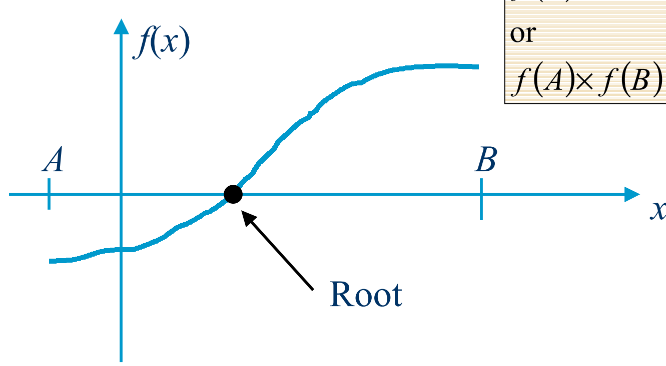
- Test for Locating a Root within Subinterval

$f(A) < 0$ and $f(B) > 0$

or

$f(A) \times f(B) = -\text{(negative)}$



# Direct-Search Method

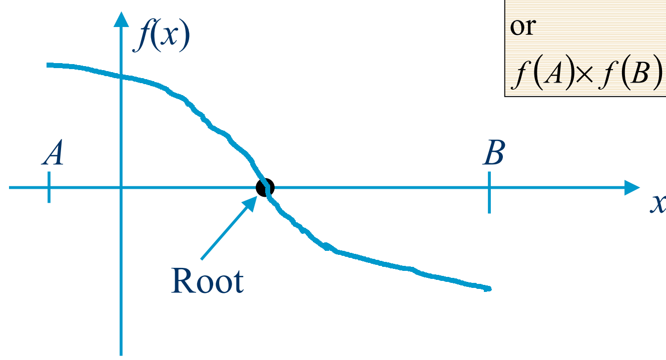


- Test for Locating a Root within Subinterval

$f(A) > 0$ and $f(B) < 0$

or

$f(A) \times f(B) = -$ (negative)






# Direct-Search Method



- Test for Locating a Root within Subinterval
  1. Alternatively stated, if the product of $f(A)$ and $f(B)$ is positive, then the function does not cross the $x$ axis and a root does not lie within the subinterval.
  2. if the product of $f(A)$ and $f(B)$ is negative, then the function crosses the $x$ axis and a root lies within the subinterval.

# Direct-Search Method



■ Example 1: Direct Search Method

Estimate the roots of the following characteristic equations using direct search method:

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188 = 0$$



# Direct-Search Method

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■ Example 1 (cont'd): Direct Search Method

| $\lambda$ | $f(\lambda)$ | $\lambda$ | $f(\lambda)$ | $\lambda$ | $f(\lambda)$ | $\lambda$ | $f(\lambda)$ |
|---|---|---|---|---|---|---|---|
| 0.00 | -0.50419 | 0.55 | 0.027717 | 1.10 | -0.25713 | 1.65 | -0.36047 |
| 0.05 | -0.39583 | 0.60 | 0.020572 | 1.15 | -0.28902 | 1.70 | -0.32637 |
| 0.10 | -0.30173 | 0.65 | 0.007427 | 1.20 | -0.31867 | 1.75 | -0.28176 |
| 0.15 | -0.22112 | 0.70 | -0.01097 | 1.25 | -0.34531 | 1.80 | -0.22591 |
| 0.20 | -0.15327 | 0.75 | -0.03386 | 1.30 | -0.36821 | 1.85 | -0.15805 |
| 0.25 | -0.09741 | 0.80 | -0.06051 | 1.35 | -0.3866 | 1.90 | -0.07745 |
| 0.30 | -0.05281 | 0.85 | -0.09015 | 1.40 | -0.39975 | 1.95 | 0.016657 |
| 0.35 | -0.0187 | 0.90 | -0.12205 | 1.45 | -0.40689 | 2.00 | 0.125012 |
| 0.40 | 0.005652 | 0.95 | -0.15544 | 1.50 | -0.40729 | 2.05 | 0.248367 |
| 0.45 | 0.021007 | 1.00 | -0.18959 | 1.55 | -0.40018 | 2.10 | 0.387472 |
| 0.50 | 0.028112 | 1.05 | -0.22373 | 1.60 | -0.38483 | 2.15 | 0.543077 |

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188$$

# Direct-Search Method

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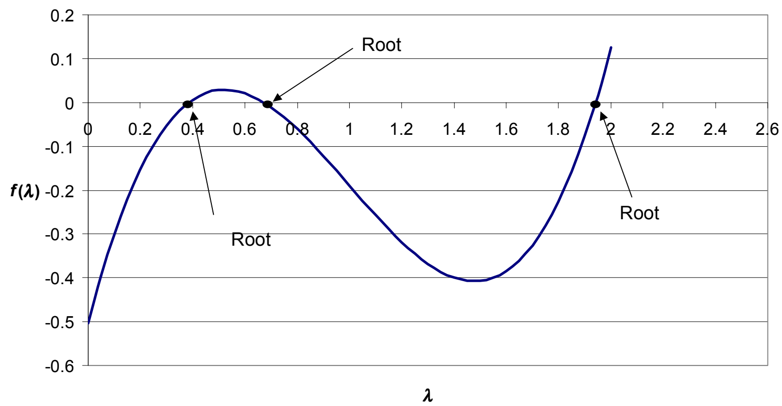
■ Example 1 (cont'd): Direct Search Method




$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188$

$\lambda_1 \approx 0.4$, $\lambda_2 \approx 0.7$, and $\lambda_3 \approx 1.9$





# Direct-Search Method

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■ Example 2: Direct-Search Method

In the previous example (Example 1) if an accuracy (tolerance) of 0.005 is required what will be the resulting roots?

$$f(\lambda) = \lambda^3 - 3\lambda^2 + 2.3146\lambda - 0.504188 = 0$$

Consider the following subintervals

$[0.35, 0.40]$ $[0.65, 0.70]$ $[1.90, 1.95]$

# Direct-Search Method

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■ Example 2 (cont'd): Direct-Search Method

| $\lambda$ | $f(\lambda)$ |
|---|---|
| 0.3500 | -0.0187 |
| 0.3550 | -0.0158 |
| 0.3600 | -0.0131 |
| 0.3650 | -0.0104 |
| 0.3700 | -0.0078 |
| 0.3750 | -0.0054 |
| 0.3800 | -0.0030 |
| 0.3850 | -0.0007 |
| 0.3900 | 0.0015 |
| 0.3950 | 0.0036 |
| 0.4000 | 0.0057 |

Interval $[0.35, 0.40]$:

$$\text{Root} = \frac{0.3850 + 0.3900}{2}$$

$$\text{Root} = 0.3875 \approx 0.388$$




# Direct-Search Method

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■ Example 2 (cont'd): Direct-Search Method

| $\lambda$ | $f(\lambda)$ |
|---|---|
| 0.6500 | 0.0074 |
| 0.6550 | 0.0058 |
| 0.6600 | 0.0041 |
| 0.6650 | 0.0024 |
| 0.6700 | 0.0007 |
| 0.6750 | -0.0012 |
| 0.6800 | -0.0030 |
| 0.6850 | -0.0049 |
| 0.6900 | -0.0069 |
| 0.6950 | -0.0089 |
| 0.7000 | -0.0110 |

Interval $[0.65, 0.70]$:

$$\text{Root} = \frac{0.6700 + 0.6750}{2}$$

$$\text{Root} = 0.6725 \approx 0.673$$

# Direct-Search Method

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- Example 2 (cont'd): Direct-Search Method

| $\lambda$ | $f(\lambda)$ |
|---|---|
| 1.9000 | -0.0774 |
| 1.9050 | -0.0687 |
| 1.9100 | -0.0597 |
| 1.9150 | -0.0507 |
| 1.9200 | -0.0415 |
| 1.9250 | -0.0321 |
| 1.9300 | -0.0227 |
| 1.9350 | -0.0130 |
| 1.9400 | -0.0033 |
| 1.9450 | 0.0066 |
| 1.9500 | 0.0167 |

$$\text{Interval}[1.90, 1.95]:$$

$$\text{Root} = \frac{1.9400 + 1.9450}{2}$$

$$\text{Root} = \approx 1.9425 = 1.943$$



# Direct-Search Method

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- Comparison of the Results of Examples 1 and 2

| Root | Example 1 | Example 2 |
|---|---|---|
| 1 | 0.4 | 0.388 |
| 2 | 0.7 | 0.673 |
| 3 | 1.9 | 1.943 |

# Direct-Search Method



- Example 3: Direct-Search Method

  Using the direct-search method, find the roots of the following polynomial:

$$f(x) = x^3 - 7x^2 + 14x - 8$$

  Assume a tolerance of 10%.



# Direct-Search Method

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- Example 3: Direct-Search Method

| *x* | *f(x)* | *x* | *f(x)* |
|---|---|---|---|
| 0 | -8 | 2.7 | -1.547 |
| 0.3 | -4.403 | 2.9 | -1.881 |
| 0.5 | -2.625 | 3.1 | -2.079 |
| 0.7 | -1.287 | 3.3 | -2.093 |
| 0.9 | -0.341 | 3.5 | -1.875 |
| 1.1 | 0.261 | 3.7 | -1.377 |
| 1.3 | 0.567 | 3.9 | -0.551 |
| 1.5 | 0.625 | 4.1 | 0.651 |
| 1.7 | 0.483 | 4.3 | 2.277 |
| 1.9 | 0.189 | 4.5 | 4.375 |
| 2.1 | -0.209 | 4.7 | 6.993 |
| 2.3 | -0.663 | 4.9 | 10.179 |
| 2.5 | -1.125 | | |

The roots are:
1, 2, and 4

# Direct-Search Method



- General Characteristics of the Direct-Search Method
  - The direct-search method can find the roots of any function as long as the roots are *real* and within the specified interval.
  - For a high degree of precision,the method can be cumbersome.
  - In this case, the subinterval size must be very small and a large number of calculations must be performed.





# Direct-Search Method

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- General Characteristics of the Direct-Search Method
  - To minimize the number of calculations, the subintervals size must be increased; but this obviously will reduce the precision of the estimate of the root.
  - Caution must taken when selecting the size of the subintervals.

# Direct-Search Method



- General Characteristics of the Direct-Search Method
  - If the subinterval size is so large, some roots might be missed entirely by applying the direct-search method.
  - The interval size could be so large that more than one root occurs in a single subinterval.





# Direct-Search Method

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- General Characteristics of the Direct-Search Method
  - The direct-search method is the most straightforward technique for determining all roots within a given interval.
  - The method assumes that there is one and only one root within each subinterval.

# Direct-Search Method



- General Characteristics of the Direct-Search Method
  - If there is an even number of roots within the search interval, then $f(A)$ and $f(B)$ will have the same sign, and the search process will miss the roots within the interval. For example, consider

$$x^3 - 1.5x^2 - 0.52x + 1.056 = 0$$





# Direct-Search Method

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- General Characteristics of the Direct-Search Method

$$f(x) = (x - 1.1)(x - 1.2)(x + 0.8) = 0$$

If the search interval (subinterval size) is 0.25, with $A$ = 0.1 and $B$ = 1.25, then $f(A)$ = 0.036 and $f(B)$ = 0.0154.

In this case, the search will proceed to the next search interval and miss the roots $x$ = 1.1 and $x$ = 1.2.

# Direct-Search Method

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- Two Major Drawbacks with the Direct-search Method
  1) Multiple Roots
     - The direct search will miss the multiple roots because it will not identify any roots regardless of the size of the search interval.
  2) Point of Discontinuity
     - Another case that might result in the failure of the method in finding the roots is the case where the function has discontinuity at a point within the interval [*A*, *B*].





# Direct-Search Method

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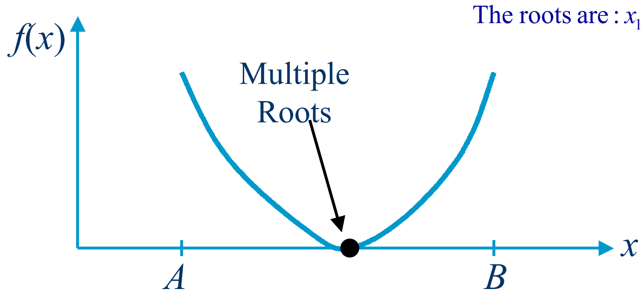
- Two Major Drawbacks with the Direct-search Method
  – Multiple Roots

$$f(x) = x^2 - 2x + 1 = (x-1)^2 = 0$$

The roots are: $x_1 = 1$ and $x_2 = 1$



# Direct-Search Method



- Two Major Drawbacks with the Direct-search Method
  - Point of Discontinuity
    - Example

$$f(x) = x^2 - \cos(x) + e^{x^{-2}} - 233$$



# Direct-Search Method



| x | f(x) | x | f(x) | x | f(x) |
|---|---|---|---|---|---|
| -30 | 667.8469 | -6 | -196.932 | 15 | -6.23586 |
| -29 | 609.7492 | -5 | -207.243 | 16 | 24.96157 |
| -28 | 552.9639 | -4 | -215.282 | 17 | 57.27863 |
| -27 | 497.2935 | -3 | -221.892 | 18 | 91.34277 |
| -26 | 443.3546 | -2 | -227.3 | 19 | 128.0141 |
| -25 | 392.0104 | -1 | -229.822 | 20 | 167.5944 |
| -24 | 343.5776 | -0.5 | -179.029 | 21 | 209.55 |
| -23 | 297.5347 | -0.3 | 66676.63 | 22 | 253.002 |
| -22 | 253.002 | 0 | $\infty$ | 23 | 297.5347 |
| -21 | 209.55 | 0.5 | -179.029 | 24 | 343.5776 |
| -20 | 167.5944 | 1 | -229.822 | 25 | 392.0104 |
| -19 | 128.0141 | 2 | -227.3 | 26 | 443.3546 |
| -18 | 91.34277 | 3 | -221.892 | 27 | 497.2935 |
| -17 | 57.27863 | 4 | -215.282 | 28 | 552.9639 |
| -16 | 24.96157 | 5 | -207.243 | 29 | 609.7492 |
| -15 | -6.23586 | 6 | -196.932 | 30 | 667.8469 |
| -14 | -36.1316 | 7 | -183.733 | | |
| -13 | -63.9015 | 8 | -167.839 | | |
| -12 | -88.8369 | 9 | -150.076 | | |
| -11 | -110.996 | 10 | -131.151 | | |
| -10 | -131.151 | 11 | -110.996 | | |
| -9 | -150.076 | 12 | -88.8369 | | |
| -8 | -167.839 | 13 | -63.9015 | | |
| -7 | -183.733 | 14 | -36.1316 | | |

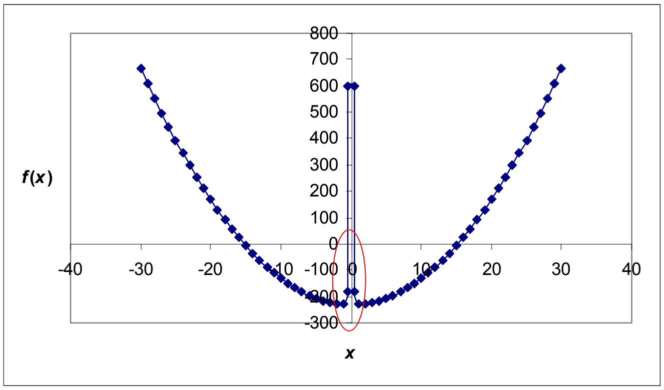
Point of Discontinuity

$$f(x) = x^2 - \cos(x) + e^{x^{-2}} - 233$$

# Direct-Search Method



– Point of Discontinuity



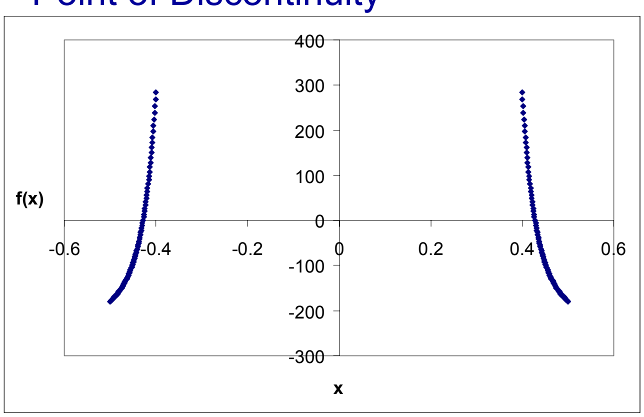

$$f(x) = x^2 - \cos(x) + e^{x^{-2}} - 233$$



# Direct-Search Method



– Point of Discontinuity




$$f(x) = x^2 - \cos(x) + e^{x^{-2}} - 233$$