CHAPTER 3b. INTRODUCTION TO NUMERICAL METHODS

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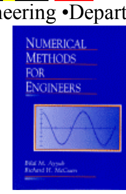

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering

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Significant Figures

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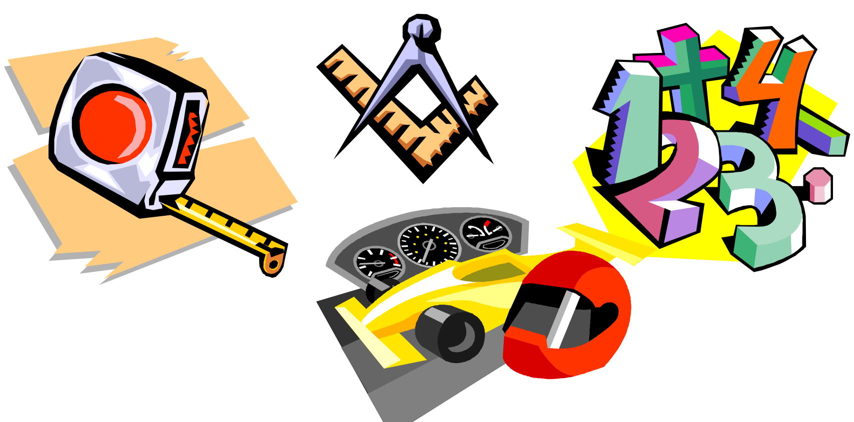
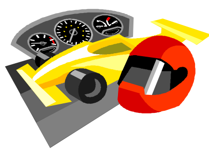
- Confidence in Measurements



Significant Figures

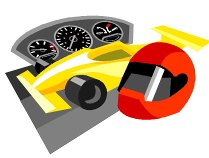
- Confidence in Measurements
  - Number Representation



- Whenever a number is employed in a computation, we must have assurance that it can be used with confidence.
- Visual inspection a car speedometer might indicate that the car is traveling between 58 and 59 mph. If the indicator is higher than the midpoint between the marker on the gauge, we can say with assurance that car is traveling at approximately 59 mph.

Significant Figures

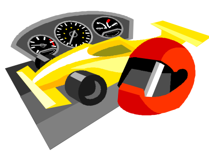
- Confidence in Measurements
  - Number Representation



- We have confidence in this result because two or more reasonable individuals reading this gauge would come to the same conclusion.
- However, let's say that we insist the speed be estimated to one decimal place. For this case, one person might say 58.8, whereas another might say 58.9 mph.

Significant Figures

- Confidence in Measurements
  - Number Representation
    - Therefore, because of the limits of this speedometer, only the first digit can be used with *confidence*.
    - Estimates of the third digit (or higher) must be viewed as approximations.
    - It would be ludicrous to claim, on the basis of this speedometer, that the car is traveling at 58.864345 mph.



Significant Figures

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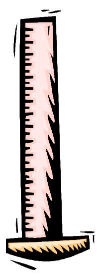
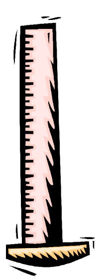
- Significant Digits
  - The ***significant digits*** of a number are those that can be used with confidence.
  - They correspond to the number of certain digits plus one estimated digit.

Significant Figures



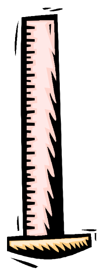
- Example 1:
  - Consider the problem of measuring the distance between two points using a ruler that has a scale with 1 mm between the finest divisions.
  - If we record our measurements in centimeter and if we estimate fractions of a millimeter, then a distance recorded as 3.76 cm gives two precise digits (3 and 7).

Significant Figures



- Example 1 (cont'd):
  - If we define a significant digit to be any number that is relatively precise, then the measurement of 3.76 cm has three significant digits.
  - Even though the last digit could be a 5 or a 7, it still provides some information about the length, and so it is considered significant.

Significant Figures



- Example 1 (cont'd):
  - If we recorded the number as 3.762, we would still have only three significant digits since the 2 is not precise.
  - Only one imprecise digit can be considered as a significant digit.

Significant Figures

- Example 2: Digital Bathroom Scale
  - A digital bathroom scale that shows weight to the nearest pound (lb) uses up to three significant digits.
  - If the scale shows, for example, 159 pounds, the the individual assumes his or her weight is within 0.5 pound of the observed value.
  - In this case, the scale has set the number of significant digits.

Significant Figures

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- Rule for Significant Digits

  The digits 1 to 9 are always significant, with zero being significant when it is not being used to set the position of the decimal point.

Significant Figures

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- Examples: Significant Digits

  2,410

  2.41

  0.00241

  – Each of the above numbers has *three significant digits*.

  – In the number 2,410, the zero (0) is used to set the decimal place.

Significant Figures

- Examples: Significant Digits
  - Confusion can be avoided by using scientific notation, for example

    2.41×10^3 means it has *three* significant digits

    2.410×10^4 means it has *four* significant digits

    2.4100×10^4 means it has *five* significant digits

Significant Figures

- Examples: Significant Digits
  - The numbers

    18, 18.00, and 18.000

    differ in that the first is recorded at two significant digits, while the second and third are recorded at four and five significant digits, respectively.

Significant Figures

- Rule for Setting Significant Digits when Performing Calculations

  Any mathematical operation using an imprecise digit is imprecise.

Significant Figures

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- Example: Arithmetic Operations and Significant Digits
  - Consider the following multiplication of two numbers:

    4.2<u>6</u> and 8.3<u>9</u>

    Each of these number has three significant digits with the last digit of each being imprecise.

Significant Figures

- Example (cont'd): Arithmetic Operations and Significant Digits

4.26	Starting number
8.39	Starting number
0.3834	0.09 times 4.26
1.278	0.3 times 4.26
34.08	
35.7414	Total (product result)

Significant Figures

- Example (cont'd): Arithmetic Operations and Significant Digits
  - The digits that depend on imprecise digits are underlined. In the final answer, only the first digits (35) are not based on imprecise digits.
  - Since one and only one imprecise digit can be considered as significant, then the result should be recorded as 35.7

Analysis of Numerical Errors

- Types of Errors
  – An error in estimating or determining a quantity of interest can be defined as a deviation from its unknown true value.
  – Errors can be classified as
    1. Non-numerical Errors
    2. Numerical Errors

Analysis of Numerical Errors

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- Non-numerical Errors
  – Modeling errors
  – Blunders and mistakes
  – Uncertainty in information and data
- Numerical Errors
  – Round-off errors
  – Truncation errors
  – Propagation errors
  – Mathematical-approximation errors

Analysis of Numerical Errors

■ Measurement and Truncation Errors

– The *error*, designated as *e*, can be defined as

$$e = x_c - x_t \qquad (1)$$

The *relative error*, denoted as e_r, is defined as

$$e_r = \frac{x_c - x_t}{x_t} = \frac{e}{x_t} \qquad (2)$$

where

x_c = computed value and x_t = true value.

Analysis of Numerical Errors

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■ Measurement and Truncation Errors

– The relative error e_r can also be expressed as percentage as

$$e_r = \frac{x_c - x_t}{x_t} \times 100\%$$

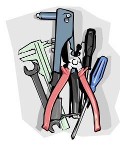
or

$$\text{Absolute } e_r = \text{ABS}\, e_r = \left| \frac{x_c - x_t}{x_t} \right| \times 100\%$$

Analysis of Numerical Errors

- Example: Measurement & Errors

The lengths of a bridge and a rivet were measured to have values of 9999 and 9 cm, respectively. If the true value values are 10,000 and 10 cm, respectively, compute (a) the absolute error and (b) the absolute relative error for each case.



Analysis of Numerical Errors

- Example (cont'd): Measurement & Errors

(a) Absolute error

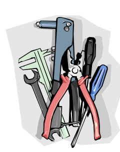
$$\text{Bridge}: e = \left|x_c - x_t\right| = \left|9999 - 10000\right| = 1\ \text{cm}$$

$$\text{Rivet}: \ e = \left|10 - 9\right| = 1\ \text{cm}$$

(b) Absolute relative error

$$\text{Bridge}: \ e_r = \left|\frac{x_c - x_t}{x_t}\right| \times 100 = \left|\frac{9999 - 10000}{10000}\right| \times 100 = 0.01\%$$

$$\text{Rivet}: \ e_r = \left|\frac{9 - 10}{10}\right| \times 100 = 10\%$$



Analysis of Numerical Errors

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- Errors in Numerical Solutions
  - In real situations, the true value is not known, so the previous equations (Eqs. 1 and 2) cannot be used to compute the errors.
  - In such cases, the best estimate of the number x should be used.
  - Unfortunately, the best estimate is the computed estimate.

Analysis of Numerical Errors

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- Errors in Numerical Solutions
  - If Eq. 1 is used iteratively, then

$$e_i = x_i + x_t \qquad (3)$$

where e_i = error in the x at iteration i, and x_i is the computed value of x from iteration i.

  - Similarly, the error for iteration $i + 1$is

$$e_{i+1} = x_{i+1} - x_t \qquad (4)$$

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- Errors in Numerical Solutions
  - Therefore, the change in the error Δe_i can be computed using Eqs. 3 and 4 as

$$\Delta e_i = e_{i+1} - e_i = x_{i+1} - x_t - \left(x_i - x_t\right)$$
$$= x_{i+1} - x_i$$

  - It can be shown that e_{i+1} is expected to be smaller than Δe_i, so if the iteration is continued until Δe is smaller than a tolerable error, then x_{i+1} will be sufficiently close to x_i

Analysis of Numerical Errors

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- Errors in Numerical Solutions

$$\Delta e_i = x_{i+1} - x_i$$

$$\left(\Delta e_i\right)_r = \frac{x_{i+1} - x_i}{x_{i+1}}$$

$$\text{ABS}\left(\Delta e_i\right)_r = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Analysis of Numerical Errors

■ Example: Root of a Polynomial

$$x^3 - 3x^2 - 6x + 8 = 0$$

- Dividing both sides of the equation by *x,* yields

$$x^2 - 3x - 6 + \frac{8}{x} = 0$$

- Solving for *x* using the x^2 term, gives

$$x = \sqrt{3x + 6 - \frac{8}{x}}$$

Analysis of Numerical Errors

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■ <u>Example (cont'd):</u> Root of a Polynomial

Last Eq. can be solved iteratively as follows:

$$x_{i+1} = \sqrt{3x_i + 6 - \frac{8}{x_i}}$$

– If an initial value of 2 (x_0 = 2) is assumed for *x,* then

$$x_1 = \sqrt{3x_0 + 6 - \frac{8}{x_0}} = \sqrt{3(2) + 6 - \frac{8}{2}} = 2.828427$$

Analysis of Numerical Errors

- Example (cont'd): Root of a Polynomial
  - Now x_1 = 2.828427
  - A second iteration will yield

$$x_2 = \sqrt{3x_1 + 6 - \frac{8}{x_1}} = \sqrt{3(2.828427) + 6 - \frac{8}{2.828427}} = 3.414213$$

  - A third iteration results in

$$x_3 = \sqrt{3x_2 + 6 - \frac{8}{x_2}} = \sqrt{3(3.414213) + 6 - \frac{8}{3.414213}} = 3.728202$$

Analysis of Numerical Errors

- Example (cont'd): Root of a Polynomial
  - Therefore,

$$\Delta e_i = x_{i+1} - x_i$$
$$\Delta e_1 = x_1 - x_0 = 2.828427 - 2.000000 = 0.826427$$
$$\Delta e_2 = x_2 - x_1 = 3.414213 - 2.828427 = 0.585786$$
$$\Delta e_3 = x_3 - x_2 = 3.728202 - 3.414213 = 0.313989$$

  - The results of 10 iteration are shown the table of the next viewgraph.

Analysis of Numerical Errors

■ Example (cont'd): Root of a Polynomial

i	x_i	Δe_i	$\lvert(\Delta e_i)_r\rvert$%
0	2.000000	-	-
1	2.828427	0.828427	29.29
2	3.414214	0.585786	17.16
3	3.728203	0.313989	8.42
4	3.877989	0.149787	3.86
5	3.946016	0.068027	1.72
6	3.976265	0.030249	0.76
7	3.989594	0.013328	0.33
8	3.995443	0.005849	0.15
9	3.998005	0.002563	0.06
10	3.999127	0.001122	0.03