

# CHAPTER 2c. MATRICES



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## Matrix Operations



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### ■ Matrix Inversion

- The inverse of an  $n \times n$  matrix  $A$  is an  $n \times n$  matrix  $B$  having the property that

$$AB = BA = I$$

- $B$  is called the inverse of  $A$  and is usually denoted by  $A^{-1}$ .

- Hence,

$$A^{-1} A = I$$



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## ■ Matrix Inversion

### – Properties:

- If a square matrix **A** has an inverse, it is said to be *invertible* or *nonsingular*.
- If it does not possess an inverse, it is *singular*.
- In particular, the identity or unit matrix **I** is invertible and is its own inverse since

$$II = I$$



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## ■ Matrix Inversion

- The inverse can be determined by forming  $n^2$  simultaneous equations and solving for  $n^2$  unknowns.
- Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$



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## Matrix Inversion

– Therefore,

$$A^{-1}A = I$$

$$A^{-1}A = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & c_{11}a_{12} + c_{12}a_{22} \\ c_{21}a_{11} + c_{22}a_{21} & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix}$$

or

$$\begin{bmatrix} c_{11}a_{11} + c_{12}a_{21} & c_{11}a_{12} + c_{12}a_{22} \\ c_{21}a_{11} + c_{22}a_{21} & c_{21}a_{12} + c_{22}a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



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## Matrix Inversion

– Hence, the following for simultaneous equations can be solved for  $c_{ij}$  given the values  $a_{ij}$  of the original matrix  $A$ :

$$c_{11}a_{11} + c_{12}a_{21} = 1$$

$$c_{11}a_{12} + c_{12}a_{22} = 0$$

$$c_{21}a_{11} + c_{22}a_{21} = 0$$

$$c_{21}a_{12} + c_{22}a_{22} = 1$$



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## ■ Example 1: Matrix Inversion

$$\text{Find } \mathbf{A}^{-1} \text{ if } \mathbf{A} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2c_{11} + 5c_{12} & 3c_{11} + 7c_{12} \\ 2c_{21} + 5c_{22} & 3c_{21} + 7c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



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## ■ Example 1 (cont'd): Matrix Inversion

$$2c_{11} + 5c_{12} = 1$$

$$3c_{11} + 7c_{12} = 0$$

$$2c_{21} + 5c_{22} = 0$$

$$3c_{21} + 7c_{22} = 1$$

From which,

$$c_{11} = -7, c_{12} = 3, c_{21} = 5, \text{ and } c_{22} = -2$$

$$\text{Therefore, } \mathbf{A}^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$



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## ■ Example 2: Matrix Inversion

$$\text{Find } \mathbf{A}^{-1} \text{ if } \mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3c_{11} - 2c_{12} + c_{13} & c_{11} + 5c_{12} + 3c_{13} & 2c_{11} + 4c_{12} + 6c_{13} \\ 3c_{21} - 2c_{22} + c_{23} & c_{21} + 5c_{22} + 3c_{23} & 2c_{21} + 4c_{22} + 6c_{23} \\ 3c_{31} - 2c_{32} + c_{33} & c_{31} + 5c_{32} + 3c_{33} & 2c_{31} + 4c_{32} + 6c_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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## ■ Example 2 (cont'd): Matrix Inversion

$$3c_{11} - 2c_{12} + c_{13} = 1$$

$$c_{11} + 5c_{12} + 3c_{13} = 0$$

$$2c_{11} + 4c_{12} + 6c_{13} = 0$$

$$3c_{21} - 2c_{22} + c_{23} = 0$$

$$c_{21} + 5c_{22} + 3c_{23} = 1$$

$$2c_{21} + 4c_{22} + 6c_{23} = 0$$

$$3c_{31} - 2c_{32} + c_{33} = 0$$

$$c_{31} + 5c_{32} + 3c_{33} = 0$$

$$2c_{31} + 4c_{32} + 6c_{33} = 1$$

3 Sets of Simultaneous Equations



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## ■ Example 2 (cont'd): Matrix Inversion

$$\begin{aligned} 3c_{11} - 2c_{12} + c_{13} &= 1 & c_{11} &= 0.3750 \\ c_{11} + 5c_{12} + 3c_{13} &= 0 & \implies c_{12} &= 0 \\ 2c_{11} + 4c_{12} + 6c_{13} &= 0 & c_{13} &= -0.1250 \end{aligned}$$

$$\begin{aligned} 3c_{21} - 2c_{22} + c_{23} &= 0 & c_{21} &= 0.3333 \\ c_{21} + 5c_{22} + 3c_{23} &= 1 & \implies c_{22} &= 0.3333 \\ 2c_{21} + 4c_{22} + 6c_{23} &= 0 & c_{23} &= -0.3333 \end{aligned}$$



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## ■ Example 2 (cont'd): Matrix Inversion

$$\begin{aligned} 3c_{31} - 2c_{32} + c_{33} &= 0 & c_{31} &= -0.2292 \\ c_{31} + 5c_{32} + 3c_{33} &= 0 & \implies c_{32} &= -0.1667 \\ 2c_{31} + 4c_{32} + 6c_{33} &= 1 & c_{33} &= 0.3542 \end{aligned}$$

Therefore,

$$A^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0.3750 & 0 & -0.1250 \\ 0.3333 & 0.3333 & -0.3333 \\ -0.2292 & -0.1667 & 0.3542 \end{bmatrix}$$

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## ■ Matrix Singularity

- If the inverse of a matrix exists, then the matrix is **nonsingular**.
- If the inverse does not exist, then the matrix is **singular**.

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## ■ Matrix Singularity

- Matrix Singularity and System of Equations
  - One implication of matrix singularity in solving a system of simultaneous equations is that a *unique solution* for the equation does not exist.
  - If the matrix is singular, the system of equation will not have a solution.



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## ■ Matrix Singularity

### – Matrix Singularity and System of Equations

$$2X_1 + 3X_2 = a$$

$$4X_1 + 6X_2 = b$$

If  $2a = b$ , then there are an infinite number of solutions.

For example, three possibilities are:

1)  $X_1 = 2, X_2 = 1, a = 7$ , and  $b = 14$

2)  $X_1 = -1, X_2 = 4, a = 10$ , and  $b = 20$

3)  $X_1 = 0, X_2 = -2, a = -6$ , and  $b = -12$



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## ■ Matrix Singularity

### – Matrix Singularity and System of Equations

$$2X_1 + 3X_2 = a$$

$$4X_1 + 6X_2 = b$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

If  $a \neq b$ , then there is no feasible solution.

For example, if  $a = 2$  and  $b = 3$ , there are no values of  $X_1$  and  $X_2$  that can satisfy the equality of the two equations.





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## ■ Trace of a Matrix

- The trace of square matrix is the sum of the diagonal elements as defined by

$$tr(A) = \sum_{i=1}^n a_{ii}$$



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## ■ Trace of a Matrix

- Example:

Find the trace of the following matrices:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 5 & 4 \\ 1 & 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 11 & 3 \\ -5 & -5 \end{bmatrix}$$

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$$tr(A) = 3 + 5 + 6 = 14$$

$$tr(B) = 11 + (-5) = 6$$

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## ■ Matrix Augmentation

– Matrix augmentation is the addition of a column or columns to the initial matrix

$$\left[ \begin{array}{cccc|cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1c} & c_{11} & c_{12} & c_{13} & \dots & c_{1c} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2c} & c_{21} & c_{22} & c_{23} & \dots & c_{2c} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3c} & c_{31} & c_{32} & c_{33} & \dots & c_{3c} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \dots & a_{rc} & c_{r1} & c_{r2} & c_{r3} & \dots & c_{rc} \end{array} \right]$$

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## ■ Examples: Matrix Augmentation

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 4 & 1 \\ 2 & 3 & 10 \end{bmatrix}$$

$$A_a = \left[ \begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array} \right]$$

$$B_a = \left[ \begin{array}{ccc|c} 0 & 3 & 4 & 2 \\ 1 & 4 & 1 & 3 \\ 2 & 3 & 10 & 6 \end{array} \right]$$



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## ■ Submatrices and Partitioning

– Given any matrix  $A$ , a submatrix of  $A$  is a matrix obtained from  $A$  by removing any number of rows or columns.

– Thus if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad C = [2 \quad 3 \quad 4]$$



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## ■ Submatrices and Partitioning

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix}, \quad \text{and} \quad C = [2 \quad 3 \quad 4]$$

- Then  $B$  and  $C$  are both submatrices of  $A$
- $B$  is obtained by removing from  $A$  the first and second rows together with the first and third columns
- $C$  is obtained by removing from  $A$  the second, third, and fourth rows together with the first column.



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## ■ Submatrices and Partitioning

- A matrix can be partitioned by separating it into smaller matrices.
- For example, matrix **A** can be partitioned into four other matrices as

$$A = \left[ \begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right]$$



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## ■ Example: Matrix Partition

$$A = \begin{bmatrix} 1 & 0.2 & 0.9 \\ 0.2 & 1 & 0.8 \\ 0.9 & 0.8 & 1 \end{bmatrix} \Rightarrow A = \left[ \begin{array}{cc|c} 1 & 0.2 & 0.9 \\ 0.2 & 1 & 0.8 \\ \hline 0.9 & 0.8 & 1 \end{array} \right]$$

$$A_{11} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0.9 \\ 0.8 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0.9 & 0.8 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 \end{bmatrix}$$

# Vectors



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## ■ Definitions

- A vector is a  $1 \times n$  or  $n \times 1$  matrix
- A  $1 \times n$  matrix is called a row vector
- An  $n \times 1$  matrix is called a column vector
- The elements are called the components of the vector.
- The number of components, in this case, is its dimension.

# Vectors



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## ■ Examples: Vectors

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$V_2 = [t \quad 2t \quad -t \quad 0]$$

$$Z_1 = [-0.5 \quad 4] \quad Z_2 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}$$

# Vectors



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## ■ Vectors in Space

- The name vector indicates that a one-dimensional matrix with  $n$  elements can be represented as a vector in  $n$ -dimensional space, with one end of the vector at a point and the the other end at another point.

$$V_{P_2-P_1} = \left[ \left( P_{2X_1} - P_{1X_1} \right) \quad \left( P_{2X_2} - P_{1X_2} \right) \quad \left( P_{2X_3} - P_{1X_3} \right) \quad \dots \quad \left( P_{2X_n} - P_{1X_n} \right) \right]$$

# Vectors

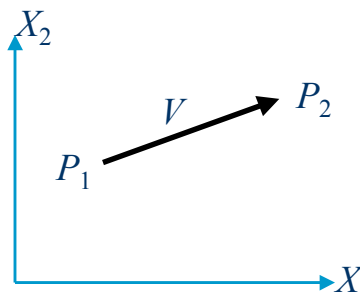


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## ■ Vector in Two-dimensional Space

- In general the vector in two-dimensional space is given by

$$V_{P_2-P_1} = \left[ \left( P_{2X_1} - P_{1X_1} \right) \quad \left( P_{2X_2} - P_{1X_2} \right) \right]$$



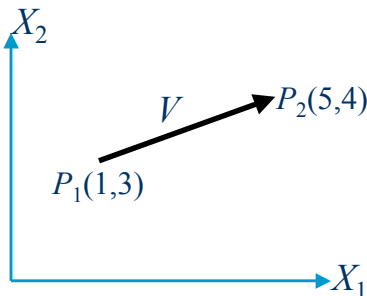
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## ■ Example: Vector in Two-dimensional Space

$$\begin{aligned} V_{P_2-P_1} &= [(P_{2x_1} - P_{1x_1}) \quad (P_{2x_2} - P_{1x_2})] \\ &= [(5-1) \quad (4-3)] \\ &= [4 \quad 1] \end{aligned}$$

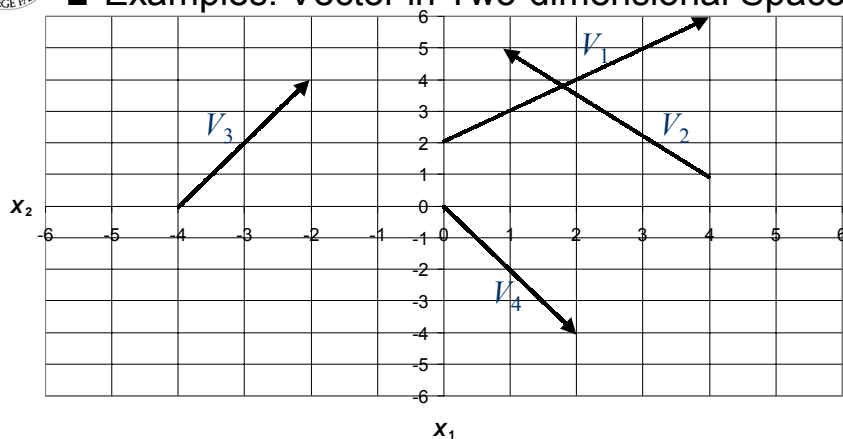


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## ■ Examples: Vector in Two-dimensional Space



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## ■ Examples: Vector in Two-dimensional Space

$$V_1 = [(4-0) \quad (6-2)] = [4 \quad 4]$$

$$V_2 = [(1-4) \quad (5-1)] = [-3 \quad 4]$$

$$V_3 = [(-2-(-4)) \quad (4-0)] = [2 \quad 4]$$

$$V_4 = [(2-0) \quad (-4-0)] = [4 \quad -4]$$

# Vectors



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## ■ Vector Operations

- The matrix operations of addition, subtraction, and multiplication can be applied to vectors.
- Two row (or column) vectors can be added or subtracted.
- A row vector with  $n$  elements can be postmultiplied by a column vector with  $n$  elements to equal a scalar value





# Vectors

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## ■ Example: Vector Operations

Find  $V_1 + 3V_2$ ,  $V_2 - 2V_1$ , and  $V_1^T V_2$  if

$$V_1 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$



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## ■ Example (cont'd): Vector Operations

$$V_1 + 3V_2 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -3 \\ 12 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 18 \\ 3 \end{bmatrix}$$

$$V_2 - 2V_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ 1 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ -8 \\ 1 \end{bmatrix}$$



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## ■ Example (cont'd): Vector Operations

$$V_1^T = [3 \quad 1 \quad 6 \quad 0]$$

$$V_1^T V_2 = [3 \quad 1 \quad 6 \quad 0] \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix} = [3(2) + 1(-1) + 6(4) + (0)(1)] = [29]$$



# Vectors

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## ■ Orthogonal and Normalized Vectors

- Two vectors are said to be orthogonal if their product equals zero.
- For the vector product  $\mathbf{A} \cdot \mathbf{B}$ ,  $\mathbf{A}$  is a row vector and  $\mathbf{B}$  is a column vector, the resulting vector product is a scalar value.
- If two vectors that are orthogonal are plotted in the  $n$ -dimensional space, the vectors will be perpendicular to each other.

# Vectors

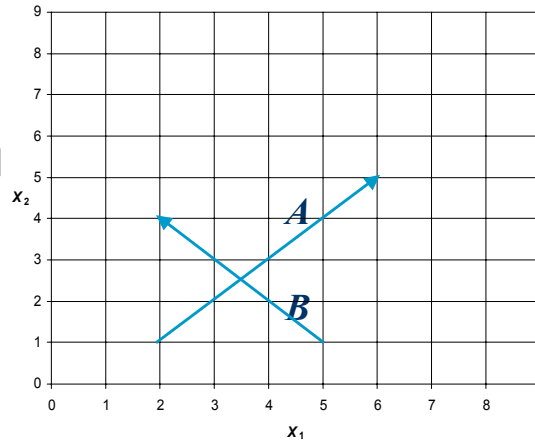


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## ■ Example: Orthogonal Vectors

$$A = [(6-2) \quad (5-1)] = [4 \quad 4]$$

$$B = [(2-5) \quad (4-1)] = [-3 \quad 3]$$



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# Vectors



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## ■ Example: Orthogonal Vectors

$$A \cdot B = [4 \quad 4] \begin{bmatrix} -3 \\ 3 \end{bmatrix} = [-12 + 12] = [0]$$

Since the vector product is zero, the vectors are perpendicular (orthogonal) to each other.

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# Vectors



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## ■ Length (magnitude) of a Vector

The length or magnitude of a vector  $V$  equals the square root of the sum of the squares of its elements, that is

$$\text{Length of } V = \|V\| = \sqrt{\sum_{i=1}^n v_i^2}$$

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## ■ Normalized Vector

- A vector is said to be normalized if each elements of the vector is divided by its length.
- A normalized vector has a length that is equal to one.
- A unit vector is also a normalized vector.

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## ■ Orthonormal Vectors

Two vectors that are both normalized and orthogonal to each other are said to be orthonormal vectors

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## ■ Example

For the following vectors  $V_1$  and  $V_2$ , perform the following:

1. Find their lengths (magnitudes)
2. Normalize them,
3. State whether they are orthonormal

$$V_1 = [2 \quad -3 \quad 5] \quad V_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

# Vectors



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## ■ Example (cont'd)

$$\text{Length of } V_1 = \sqrt{(2)^2 + (-3)^2 + (5)^2} = \sqrt{38} = 6.164$$

$$\text{Length of } V_2 = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3} = 1.732$$

The normalized vectors are:

$$V_{n1} = \begin{bmatrix} \frac{2}{\sqrt{38}} & \frac{-3}{\sqrt{38}} & \frac{5}{\sqrt{38}} \end{bmatrix}$$

$$V_{n2} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

Since the  $V_1 V_2 = 0$ ,  $V_1$  and  $V_2$  are orthogonal.  
 $V_{n1}$  and  $V_{n2}$  are orthonormal.