CHAPTER 1c: INTRODUCTION

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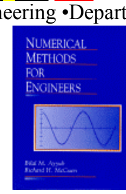

by

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ENCE 203 - Computation Methods in Civil Engineering II

Department of Civil and Environmental Engineering
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Taylor Series Expansion

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- Characteristics of Taylor Series
  - The Taylor series is of great value in the study of numerical methods.
  - In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point.
  - A Taylor series is commonly used in engineering analysis to approximate functions that do not have closed form solution.

Taylor Series Expansion

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- Characteristics of Taylor Series
  - A Taylor series is the sum of functions based on continually increasing derivatives.
  - For a function *f*(*x*) that depends on only one independent variable *x*, the value of the function at point $x_0 + h$ can be approximated by Taylor series.

Taylor Series Expansion

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- Taylor's Theorem
  - The Taylor series is given by

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)+\ldots+\frac{h^n}{n!}f^{(n)}(x_0)+R_{n+1}$$

where

x_0 = base value or starting value
x = the point at which the value of the function is needed
$h = x - x_0$ = distance between x_0 and x (step size)
$n!$ = factorial of $n = n(n-1)(n-2)\ldots 1$
$f^{(n)}$ = indicates the n^{th} derivative of the function $f(x)$
R_{n+1} = the remainder of Taylor series expansion

Taylor Series Expansion

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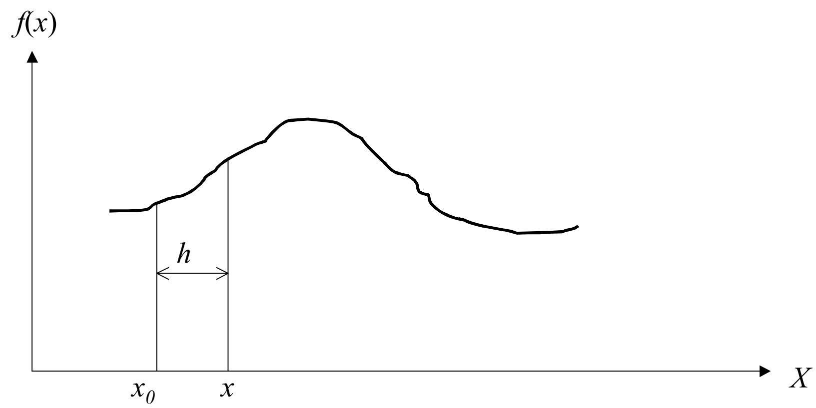

Taylor Series Expansion

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Taylor series expansion can also be given in a compact form as follows:

$$f(x_0 + h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x_0)$$

where

0! = 1 by convention

The above equation is based on the assumption that continuous derivative exist in an interval that include the points x_0 and x

Examples: Taylor Series Expansion

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- These series can be used to evaluate their corresponding functions at any point x using a base value of x0 = 0 and increment h.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + .. + \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{k=0}^{\infty} x^k$$

Examples: Taylor Series Expansion

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- Example 1:

  Show that the Taylor series expansion for the exponential function is given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^k}{k!}$$

  when $x = x_0 = 0$ as the base (or starting) point and h as the increment. Evaluate the series for h = 0.1, 0.2, 0.3, … to 1.0 using one term, two terms, and three terms. Plot your results and compare them with the true value for each case.

Examples: Taylor Series Expansion

- Example 1 (cont'd)

$$f(x) = e^x$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = e^0 = 1$$

$$f^{(2)}(x) = e^x \Rightarrow f^{(2)}(0) = e^0 = 1$$

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$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$$

Examples: Taylor Series Expansion

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- Example 1 (cont'd)

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!} f^{(2)}(x_0) + \frac{h^3}{3!} f^{(3)}(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + R_{n+1}$$

$$x_0 = 0 \Rightarrow x_0 + h = h$$

by definition, $h = x - x_0 \Rightarrow h = x$

$$f(0) = 1 = f^{(1)}(0) = f^{(2)}(0) \ldots = f^{(n)}(0)$$

Hence,

$$f(h) = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots + \frac{h^n}{n!}$$

or

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!}$$

Examples: Taylor Series Expansion

■ Example 1 (cont'd)

x	h	$f(x_0+h)$ one term	$f(x_0+h)$ two terms	$f(x_0+h)$ three terms	True value
0.1	0.1	1.000000	1.100000	1.105000	1.105171
0.2	0.2	1.000000	1.200000	1.220000	1.221403
0.3	0.3	1.000000	1.300000	1.345000	1.349859
0.4	0.4	1.000000	1.400000	1.480000	1.491825
0.5	0.5	1.000000	1.500000	1.625000	1.648721
0.6	0.6	1.000000	1.600000	1.780000	1.822119
0.7	0.7	1.000000	1.700000	1.945000	2.013753
0.8	0.8	1.000000	1.800000	2.120000	2.225541
0.9	0.9	1.000000	1.900000	2.305000	2.459603
1.0	1.0	1.000000	2.000000	2.500000	2.718282

Examples: Taylor Series Expansion

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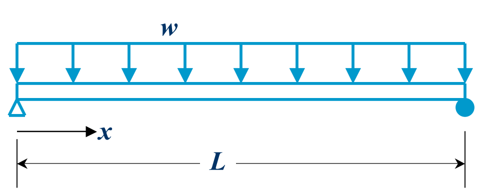
■ Example 2

The simply supported beam is subjected to uniform distributed load *w* as shown in the figure. The deflection of the beam *y* is given by

$$y = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$$



Examples: Taylor Series Expansion

■ Example 2 (cont'd)

Develop a Taylor series expansion of y for three terms (second-order approximation) using the following values:

w = 14 lb/in L = 360 in

E = 29,000,000 lb/in^2, I = 144 in^4

Tabulate and plot your results, and compare them with the true value for h = 45, 90, 135, 180, 225, and 270 in, and x_0 = 90 in.

Examples: Taylor Series Expansion

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■ Example 2 (cont'd)

$$y = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right) \Rightarrow y(90) = \frac{-14}{24(29\times10^6)166}\left[(90)^4 - 2(360)(90)^3 + (360)^3(90)\right] = -0.4532 \text{ in}$$

$$y' = -\frac{w}{24EI}\left(4x^3 - 6Lx^2 + L^3\right) \Rightarrow y'(90) = \frac{-14}{24(29\times10^6)166}\left[4(90)^3 - 6(360)(90)^2 + (360)^3\right] = -0.003887$$

$$y'' = -\frac{w}{24EI}\left(12x^2 - 12Lx\right) \Rightarrow y''(90) = \frac{-14}{24(29\times10^6)166}\left[12(90)^2 - 12(360)(90)\right] = 0.00003533$$

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

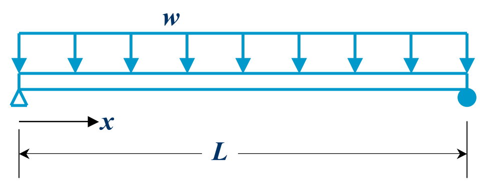
$$y(90 + h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2}$$

Examples: Taylor Series Expansion

- Example 2 (cont'd)
  - Taylor Series Approximation for the Deflection of The Beam

$$y(x) = y(x_0 + h) = y(90 + h)$$

$$y(90+h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2}$$



Examples: Taylor Series Expansion

- Example 2 (cont'd)
  - For h = 135, $x = x_0 + h$ = 90+135 = 225 in

$$y(225) = -0.4532 - 0.003887\,(135) + 0.00003533\frac{(135)^2}{2} = -0.6560$$

  - For h = 180, $x = x_0 + h$ = 90+180 = 270 in

$$y(270) = -0.4532 - 0.003887\,(180) + 0.00003533\frac{(180)^2}{2} = -0.5805$$

Examples: Taylor Series Expansion

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■ Example 2 (cont'd)

$$y(x) = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$$

x (in)	*h* (in)	Three Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.592294	-0.5888153	0.00347823
180	90	-0.659871	-0.6360199	0.02385075
225	135	-0.655896	-0.5888153	0.06708023
270	180	-0.580368	-0.4531642	0.12720399
315	225	-0.433289	-0.2469546	0.18633397
360	270	-0.214657	0	0.21465673

$$y(90+h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2}$$

Examples: Taylor Series Expansion

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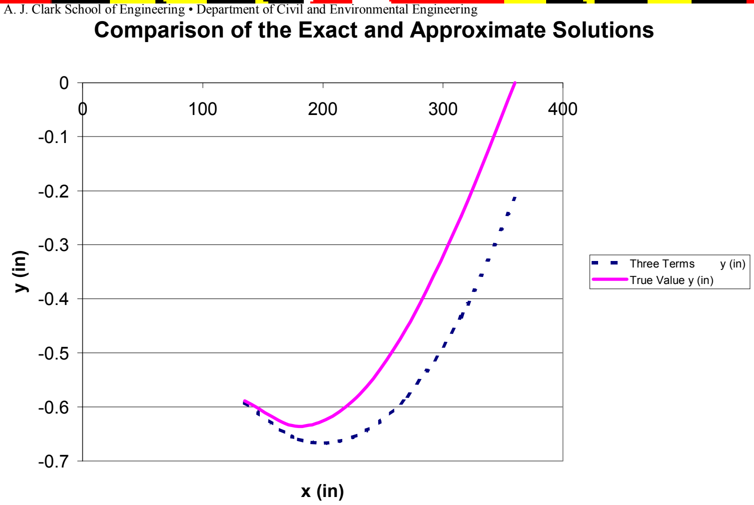
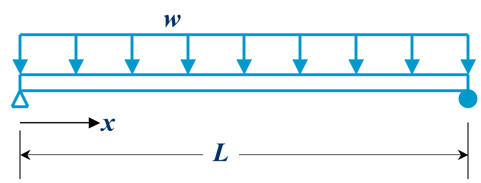


Examples: Taylor Series Expansion

- Example 3

Redo Example 2 using four and five terms, that is , use third and fourth-order approximations for the deflection *y*.

$$y = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right)$$



Examples: Taylor Series Expansion

- Example 3 (cont'd)

$$y = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3 x\right) \Rightarrow y(90) = \frac{-14}{24(29\times10^6)166}\left[(90)^4 - 2(360)(90)^3 + (360)^3(90)\right] = -0.4532 \text{ in}$$

$$y' = -\frac{w}{24EI}\left(4x^3 - 6Lx^2 + L^3\right) \Rightarrow y'(90) = \frac{-14}{24(29\times10^6)166}\left[4(90)^3 - 6(360)(90)^2 + (360)^3\right] = -0.003887$$

$$y'' = -\frac{w}{24EI}\left(12x^2 - 12Lx\right) \Rightarrow y''(90) = \frac{-14}{24(29\times10^6)166}\left[12(90)^2 - 12(360)(90)\right] = 0.00003533$$

$$y''' = -\frac{w}{24EI}\left(24x - 12L\right) \Rightarrow y'''(90) = \frac{-14}{24(29\times10^6)166}\left[24(90) - 12(360)\right] = 2.6174\times10^{-7}$$

$$y^{(4)} = -\frac{w24}{24EI} = -\frac{w}{EI} \Rightarrow y^{(4)}(90) = \frac{-14}{(29\times10^6)166} = -2.9082\times10^{-9}$$

Examples: Taylor Series Expansion

- Example 3 (cont'd)
  - Third-order Approximation (four terms)

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)+\ldots+\frac{h^n}{n!}f^{(n)}(x_0)+R_{n+1}$$

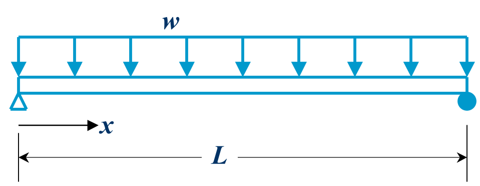
$$y(90+h)=-0.4532-0.003887h+0.00003533\frac{h^2}{2}+2.6174\times10^{-7}\frac{h^3}{6}$$

Examples: Taylor Series Expansion

- Example 3 (cont'd)
  - Third-order Approximation (four terms)

$$y(x)=y(x_0+h)=y(90+h)$$

$$y(90+h)=-0.4532-0.003887h+0.00003533\frac{h^2}{2}+2.6174\times10^{-7}\frac{h^3}{6}$$



Examples: Taylor Series Expansion

■ Example 3 (cont'd)

– For h = 135, $x = x_0 + h$ = 90+135 = 225 in

$$y(225) = -0.4532 - 0.003887\,(135) + 0.00003533\frac{(135)^2}{2} + 2.6174\times10^{-7}\,\frac{(135)^3}{6} = -0.5487$$

– For h = 180, $x = x_0 + h$ = 90+180 = 270 in

$$y(270) = -0.4532 - 0.003887\,(180) + 0.00003533\frac{(180)^2}{2} + 2.6174\times10^{-7}\,\frac{(180)^3}{6} = -0.3261$$

Examples: Taylor Series Expansion

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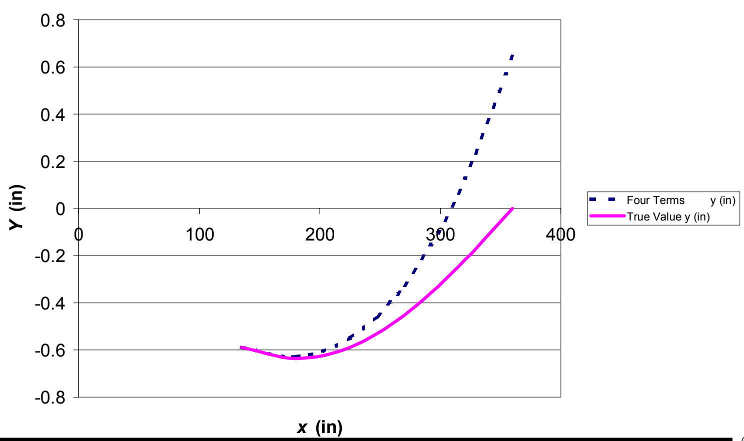
■ Example 3 (cont'd)

$$y(x) = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$$

x (in)	*h* (in)	Four Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.58832	-0.58882	0.000497
180	90	-0.62807	-0.63602	0.00795
225	135	-0.54857	-0.58882	0.040248
270	180	-0.32596	-0.45316	0.127204
315	225	0.063602	-0.24695	0.310557
360	270	0.64397	0	0.64397

$$y(90+h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2} + 2.6174\times10^{-7}\,\frac{h^3}{6}$$

Examples: Taylor Series Expansion



Examples: Taylor Series Expansion

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- Example 3 (cont'd)
  - Fourth-order Approximation (five terms)

$$f(x_0+h)=f(x_0)+hf^{(1)}(x_0)+\frac{h^2}{2!}f^{(2)}(x_0)+\frac{h^3}{3!}f^{(3)}(x_0)+\frac{h^4}{4!}f^{(4)}(x_0)+\ldots+\frac{h^n}{n!}f^{(n)}(x_0)+R_{n+1}$$

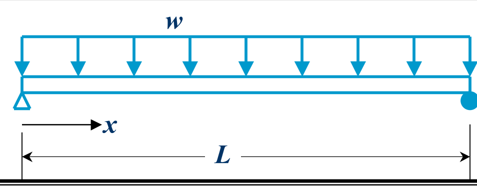
$$y(90+h)=-0.4532-0.003887h+0.00003533\frac{h^2}{2}+2.6174\times10^{-7}\frac{h^3}{6}-2.9082\times10^{-9}\frac{h^4}{24}$$

Examples: Taylor Series Expansion

- Example 3 (cont'd)
  - Fourth-order Approximation (five terms)

$$y(x) = y(x_0 + h) = y(90 + h)$$

$$y(90+h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2} + 2.6174\times10^{-7}\frac{h^3}{6} - 2.9082\times10^{-9}\frac{h^4}{24}$$



Examples: Taylor Series Expansion

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- Example 3 (cont'd)
  - For h = 135, $x = x_0 + h$ = 90+135 = 225 in

$$y(225) = -0.4532 - 0.003887\,(135) + 0.00003533\frac{(135)^2}{2} + 2.6174\times10^{-7}\frac{(135)^3}{6} - 2.9082\times10^{-9}\frac{(135)^4}{24} = -0.5889$$

  - For h = 180, $x = x_0 + h$ = 90+180 = 270 in

$$y(270) = -0.4532 - 0.003887\,(180) + 0.00003533\frac{(180)^2}{2} + 2.6174\times10^{-7}\frac{(180)^3}{6} - 2.9082\times10^{-9}\frac{(180)^4}{24} = -0.4533$$

Examples: Taylor Series Expansion

■ Example 3 (cont'd)

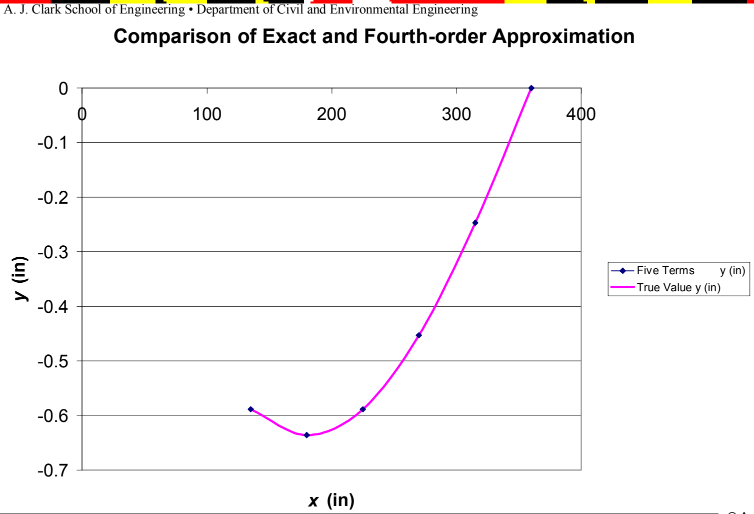
$$y(x) = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$$

x (in)	*h* (in)	Five Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.58882	-0.58882	1.11E-16
180	90	-0.63602	-0.63602	1.11E-16
225	135	-0.58882	-0.58882	1.11E-16
270	180	-0.45316	-0.45316	0
315	225	-0.24695	-0.24695	2.78E-17
360	270	0	0	0

$$y(90+h) = -0.4532 - 0.003887h + 0.00003533\frac{h^2}{2} + 2.6174\times10^{-7}\frac{h^3}{6}$$

$$-2.9082\times10^{-9}\frac{h^4}{24}$$

Examples: Taylor Series Expansion



Examples: Taylor Series Expansion

Example 2. Second-order

x (in)	*h* (in)	Three Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.592294	-0.5888153	0.00347823
180	90	-0.659871	-0.6360199	0.02385075
225	135	-0.655896	-0.5888153	0.06708023
270	180	-0.580368	-0.4531642	0.12720399
315	225	-0.433289	-0.2469546	0.18633397
360	270	-0.214657	0	0.21465673

Example 3. Third-order

x (in)	*h* (in)	Four Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.58832	-0.58882	0.000497
180	90	-0.62807	-0.63602	0.00795
225	135	-0.54857	-0.58882	0.040248
270	180	-0.32596	-0.45316	0.127204
315	225	0.063602	-0.24695	0.310557
360	270	0.64397	0	0.64397

Example 3. Fourth-order

x (in)	*h* (in)	Five Terms y (in)	True Value y (in)	Abs Error (True - App)
135	45	-0.58882	-0.58882	1.11E-16
180	90	-0.63602	-0.63602	1.11E-16
225	135	-0.58882	-0.58882	1.11E-16
270	180	-0.45316	-0.45316	0
315	225	-0.24695	-0.24695	2.78E-17
360	270	0	0	0

Examples: Taylor Series Expansion

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- Comparison

x (in)	*h* (in)	Second-order	Third-order	Fourth-order	True Value
135	45	-0.59229	-0.58832	-0.58882	-0.58882
180	90	-0.65987	-0.62807	-0.63602	-0.63602
225	135	-0.6559	-0.54857	-0.58882	-0.58882
270	180	-0.58037	-0.32596	-0.45316	-0.45316
315	225	-0.43329	0.063602	-0.24695	-0.24695
360	270	-0.21466	0.64397	0	0

Examples: Taylor Series Expansion