Homework #2
ENCE 203 - Spring 2001
Due F, 2/16

Problem1:

Textbook: 1-1

The two methods differ on the basis of their respective algorithms. The analytical method is based on analytical calculus while the numerical method is based on finite differences arithmetic. Analytical approaches provide direct solutions and will result in exact solutions if they exist. Analytical methods usually require less time to find a solution. Analytical solution procedure becomes considerably more complex when constraints are involved. Numerical analysis, on the other hand, can be used to find solutions of moderately complex problems, and it is quite easy to include constraints on the unknowns in the solutions. However, numerical methods most often require a considerable number of iterations in order to find a solution with a reasonable accuracy. The solution provided by the numerical methods is usually not exact. Therefore, error analysis and error estimations are required.

Problem 2:

Show that the Taylor series expansion for sin(x) is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$$

$x_0 = 0$, $x - x_0 = h$ ---→ $h = x$

$f(x_0) = \sin(x_0) = \sin(0) = 0$

$f'(x_0) = \cos(x_0) = \cos(0) = 1$

$f''(x_0) = -\sin(x_0) = -\sin(0) = 0$

$f'''(x_0) = -\cos(x_0) = -\cos(0) = -1$

$f^{(4)}(x_0) = \sin(x_0) = \sin(0) = 0$

$f^{(5)}(x_0) = \cos(x_0) = \cos(0) = 1$

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$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!} f^{(2)}(x_0) + \frac{h^3}{3!} f^{(3)}(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + R_{n+1}$$

$$f(h) = 0 + h + 0 - \frac{h^3}{3!} + 0 + \frac{h^5}{5!} + 0 - \frac{h^7}{7!} + \ldots.$$

or since $h = x$,

$$\sin(x) \approx f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots \quad \underline{\text{Proof}}$$

Problem 3:

Textbook: 1-2

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....$$

For $h = 0.1$

$x = x_0 + h = 0 + 0.1 = 0.1$

$\cos(0.1) \approx 1.00000000$ (one term)

$\cos(0.1) \approx 1 - \frac{(0.1)^2}{2} = 0.99500000$ (two terms)

$\cos(0.1) \approx 1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24} = 0.99500417$ (three terms)

True value = 0.99500417

The following table summarizes the results for h = 0.1 to 1.0 in an increment of 0.1:

x	h	$f(x_0+h)$ one term	$f(x_0+h)$ two terms	$f(x_0+h)$ three terms	True value
0.1	0.1	1.00000000	0.99500000	0.99500417	0.99500417
0.2	0.2	1.00000000	0.98000000	0.98006667	0.98006658
0.3	0.3	1.00000000	0.95500000	0.95533750	0.95533649
0.4	0.4	1.00000000	0.92000000	0.92106667	0.92106099
0.5	0.5	1.00000000	0.87500000	0.87760417	0.87758256
0.6	0.6	1.00000000	0.82000000	0.82540000	0.82533561
0.7	0.7	1.00000000	0.75500000	0.76500417	0.76484219
0.8	0.8	1.00000000	0.68000000	0.69706667	0.69670671
0.9	0.9	1.00000000	0.59500000	0.62233750	0.62160997
1.0	1.0	1.00000000	0.50000000	0.54166667	0.54030231

From the table above, it is clear that as the terms of the Taylor series are added incrementally, the accuracy improves as compared with the true values. This is specially true as the separation distance increases.

Problem 4:

Textbook: 1-7

See next page.

$$f(x) = x^2 - 5x^{0.5} + 6 \qquad f(2) = 2.92893$$
$$f'(x) = 2x - 2.5x^{-0.5} \qquad f'(2) = 2.23223$$
$$f''(x) = 2 + 1.25x^{-1.5} \qquad f''(2) = 2.44194$$
$$f'''(x) = -1.875x^{-2.5} \qquad f'''(2) = \text{-}0.33146$$
$$f''''(x) = 4.68750x^{-3.5} \qquad f''''(2) = 0.41432$$
$$f'''''(x) = \text{-}16.40625x^{-4.5} \qquad f'''''(2) = \text{-}0.72506$$

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$$f^{n}(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots f^{(n)}(2) = \ldots\ldots$$

Referring to Eq. 1-1 in the textbook, the value of the function, $f(x)$, can be approximated by:

$$f(x_0 + h) \approx 2.92893 + 2.23223h + 1.27097h^2 - 0.05524h^3 + 0.01726h^4$$

where $x_0 = 2$

For $h = 0.1$

$f(2.1) \approx 2.92893$ (one term)

$f(2.1) \approx 2.92893 + 2.23223(0.1) = 3.15215$ (two terms)

$f(2.1) \approx 2.92893 + 2.23223(0.1) + 1.27097(0.1)^2 = 3.16486$ (three terms)

$f(2.1) \approx 2.92893 + 2.23223(0.1) + 1.27097(0.1)^2 - 0.05524(0.1)^3 = 3.16481$ (four terms)

$f(2.1) \approx 2.92893 + 2.23223(0.1) + 1.27097(0.1)^2 - 0.05524(0.1)^3 + 0.01726(0.1)^4 = 3.16481$ (five terms)

The following table summarizes the results for h = 0.1 to 1.0 in an increment of 0.1:

x	h	$f(x_0+h)$ one term	$f(x_0+h)$ two terms	$f(x_0+h)$ three terms	$f(x_0+h)$ four terms	$f(x_0+h)$ five terms	True value
2.1	0.1	2.928930	3.152153	3.164863	3.164807	3.164809	3.164312
2.2	0.2	2.928930	3.375376	3.426215	3.425773	3.425800	3.423802
2.3	0.3	2.928930	3.598599	3.712986	3.711495	3.711635	3.707125
2.4	0.4	2.928930	3.821822	4.025177	4.021642	4.022084	4.014033
2.5	0.5	2.928930	4.045045	4.362788	4.355883	4.356961	4.344306
2.6	0.6	2.928930	4.268268	4.725817	4.713885	4.716122	4.697742
2.7	0.7	2.928930	4.491491	5.114266	5.095319	5.099463	5.074162
2.8	0.8	2.928930	4.714714	5.528135	5.499852	5.506922	5.473400
2.9	0.9	2.928930	4.937937	5.967423	5.927153	5.938477	5.895307
3.0	1.0	2.928930	5.161160	6.432130	6.376890	6.394150	6.339746

From the table above, it is clear that as the terms of the Taylor series are added incrementally, the accuracy improves as compared with the true values. This is specially true as the separation distance increases.