

Homework #10 Solution
 ENCE 203 - Spring 2001
 Due F, 5/11

Problem1:

For the following integral:

$$\int_0^3 \frac{e^x \sin x}{1+x^2} dx$$

estimate its value using:

- (a) Two and three-point Gauss Quadrature.
- (b) Simpson 1/3-rule with 5 points (four intervals).
- (c) Compare the results of part (a) and (b) with the exact solution of 2.881637.

*** SOLUTION ***

(a) Two and Three-point Gauss Quadrature:

$$\int_0^3 \frac{e^x \sin x}{1+x^2} dx \Rightarrow f(x) = \frac{e^x \sin x}{1+x^2}$$

$$x = \frac{b+a}{2} + \frac{b-a}{2} x_G \Rightarrow \frac{3+0}{2} + \frac{3-0}{2} x_G \Rightarrow 1.5 + 1.5x_G$$

$$dx = \frac{b-a}{2} dx_G \Rightarrow dx = 1.5 dx_G$$

$$\int_a^b f(x) dx \Rightarrow \int_{-1}^1 f(x_G) dx_G$$

$$\int_0^3 \frac{e^x \sin x}{1+x^2} dx = \int_{-1}^1 \frac{e^{1.5+1.5x_G} \sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2} (1.5 dx_G)$$

For two-point:

$$I = \int_{-1}^1 f(x_G) dx_G = C_1 f(x_1) + C_2 f(x_2)$$

From Gauss Quadrature Table:

$$C_1 = 1 \quad x_1 = -0.5774$$

$$C_2 = 1 \quad x_2 = 0.5774$$

$$f(x_G) = \frac{e^{1.5+1.5x_G} \sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2} (1.5)$$

$$f(-0.5774) = \frac{e^{1.5+1.5(-0.5774)} \sin[1.5+1.5(-0.5774)]}{1+[1.5+1.5(-0.5774)]^2} (1.5) = 1.194624$$

$$f(0.5774) = \frac{e^{1.5+1.5(0.5774)} \sin[1.5+1.5(0.5774)]}{1+[1.5+1.5(0.5774)]^2} (1.5) = 1.694624$$

Therefore,

$$I_{2p} = C_1 f(x_1) + C_2 f(x_2) = 1(1.194624) + 1(1.695801) = 2.890425 \leftarrow \mathbf{Ans}$$

For three-point:

$$I = \int_{-1}^1 f(x_G) dx_G = C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

From Gauss Quadrature Table:

$$C_1 = 0.5555 \quad x_1 = -0.7746$$

$$C_2 = 0.8888 \quad x_2 = 0$$

$$C_3 = 0.5555 \quad x_3 = 0.7746$$

$$f(x_G) = \frac{e^{1.5+1.5x_G} \sin(1.5+1.5x_G)}{1+(1.5+1.5x_G)^2} (1.5)$$

$$f(-0.7746) = \frac{e^{1.5+1.5(-0.7746)} \sin[1.5+1.5(-0.7746)]}{1+[1.5+1.5(-0.7746)]^2} (1.5) = 0.626123$$

$$f(0) = \frac{e^{1.5+1.5(0)} \sin[1.5+1.5(0)]}{1+[1.5+1.5(0)]^2} (1.5) = 2.063291$$

$$f(0.7746) = \frac{e^{1.5+1.5(0.7746)} \sin[1.5+1.5(0.7746)]}{1+[1.5+1.5(0.7746)]^2} (1.5) = 1.226307$$

Therefore,

$$\begin{aligned} I_{2p} &= C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) = \\ &= 0.5555(0.626123) + 0.8888(2.063291) + 0.5555(1.226307) \\ &= 2.862878 \leftarrow \mathbf{Ans} \end{aligned}$$

(b) Simpson's 1/3-rule with 5 points:

i	1	2	3	4	5
x	0	0.75	1.5	2.25	3.0
$f(x)$	0	0.923539	1.375527	1.217675	0.283447

$$I_{1/3-S_5\text{-point}} = \frac{0.75}{3} [0 + 4(1.923539) + 1.375527] + \{1.375526 + 4(1.217675) + 0.283447\}$$

$$= 2.899839 \leftarrow \mathbf{Ans}$$

(c) Comparison with the Exact Solution:

$$\text{Error } I_{G2P} = \left| \frac{2.881637 - 2.890425}{2.881637} \right| \times 100 = 0.3\%$$

$$\text{Error } I_{G3P} = \left| \frac{2.881637 - 2.862878}{2.881637} \right| \times 100 = 0.65\%$$

$$\text{Error } I_{S5P} = \left| \frac{2.881637 - 2.899839}{2.881637} \right| \times 100 = 0.63\%$$

Problem 2:

Textbook: 8-6

$$\frac{dy}{dx} = e^x + xe^x + 1$$

such that $y = 0$ at $x = 0$

$$x_0 = 0$$

$$g(x_0) = y(0) = g(0) = 0$$

$$g'(x) = \frac{dy}{dx} = e^x + xe^x + 1 \quad \Rightarrow \quad g'(0) = 2$$

$$g''(x) = \frac{d^2y}{dx^2} = 2e^x + xe^x \quad \Rightarrow \quad g''(0) = 2$$

$$g'''(x) = \frac{d^3y}{dx^3} = 3e^x + xe^x \quad \Rightarrow \quad g'''(0) = 3$$

$$g''''(x) = \frac{d^4y}{dx^4} = 4e^x + xe^x \quad \Rightarrow \quad g''''(0) = 4$$

$$g''''''(x) = \frac{d^5y}{dx^5} = 5e^x + xe^x \quad \Rightarrow \quad g''''''(0) = 5$$

Using Eq. 8-9 of the textbook, the following six-terms Taylor series expansion can be obtained:

$$g(x) = 0 + (x-0)(2) + \frac{(x-0)^2}{2!}(2) + \frac{(x-0)^3}{3!}(3) + \frac{(x-0)^4}{4!}(4) + \frac{(x-0)^5}{5!}(5)$$

For $x = 0.2$

$$g(0.2) = 0 + (0.2)(2) + \frac{(0.2)^2}{2}(2) + \frac{(0.2)^3}{6}(3) + \frac{(0.2)^4}{24}(4) + \frac{(0.2)^5}{120}(5) = 0.444280$$

The following table summarizes the estimated values of the function $y(x) = g(x)$ for the range of $x = 0$ to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	0	0	0
0.2	0.444280	0.444281	0.00012
0.4	0.996693	0.996730	0.00367
0.6	1.692840	1.693271	0.02547
0.8	2.577920	2.580433	0.09738
1	3.708333	3.718282	0.26756

Problem 3:

Textbook: 8-8

$\Delta x = h = 0.2$

x	y'	y-Euler	y-exact	error(%)
0	-1	2	2	0.00
0.2	-0.8	1.8	1.818731	-1.03
0.4	-0.64	1.64	1.67032	-1.82
0.6	-0.512	1.512	1.548812	-2.38
0.8	-0.4096	1.4096	1.449329	-2.74
1	-0.32768	1.32768	1.367879	-2.94

$y(0.0)=2$
 $y(0.2)=y(0.0)+h*y'(0.0)$
 $y(0.4)=y(0.2)+h*y'(0.2)$
 $y(0.6)=y(0.4)+h*y'(0.4)$
 $y(0.8)=y(0.6)+h*y'(0.6)$
 $y(1.0)=y(0.8)+h*y'(0.8)$

Problem 4:

Textbook: 8-21

$(1+x^2)y'-xy=0,$

$y'=xy/(1+x^2),$

$\Delta x = 0.25 = h, \quad 0.5*h = 0.125 \quad 1 \leq x \leq 2$

x	y-RK	s1	y1	s2	y2	s3	y3	s4	y-exact	error(%)
1	2	1	2.125	1.055172	2.131897	1.058597	2.264649	1.104707	2	0
1.25	2.263844	1.104314	2.401883	1.142517	2.406658	1.144789	2.550041	1.176942	2.263846	-0.00012
1.5	2.549505	1.176694	2.696592	1.203629	2.699958	1.205132	2.850788	1.228032	2.54951	-0.0002
1.75	2.850432	1.227878	3.003916	1.247301	3.006344	1.248309	3.162509	1.265004	2.850439	-0.00024
2	3.162269								3.162278	-0.00027

$y\text{-exact} = \sqrt{2*(1+x*x)}$

$error(\%) = (y\text{-RK} - y\text{-exact}) / y\text{-exact} * 100\%$

$y(1.0) = 2.0$

$s1 = f(x_i, y_i)$

$s2 = f(x_i + 0.5h, y_i + 0.5h*s1), \text{ where } y1 = y_i + 0.5h*s1$

$s3 = f(x_i + 0.5h, y_i + 0.5h*s2), \text{ where } y2 = y_i + 0.5h*s2$

$s4 = f(x_i + h, y_i + h*s3), \text{ where } y3 = y_i + h*s3$

$y_RK = y_i + (h/6)(s1 + 2*s2 + 2*s3 + s4)$

Problem 5:

Textbook: 8-33

$\frac{dy}{dx} + y - 1 = 0 \quad \text{such that } y = 2 \text{ at } x = 0$

$\frac{dy}{dx} = 1 - y$

Quadratic Model: $\hat{y} = b_0 + b_1x + b_2x^2$

Using the boundary condition

$\hat{y} = 2 = b_0 + b_1(0) + b_2(0)^2$

yields $b_0 = 2$. Thus, the quadratic model is

$$\hat{y} = 2 + b_1x + b_2x^2$$

and

$$\frac{d\hat{y}}{dx} = b_1 + 2b_2x$$

$$\begin{aligned} e &= \frac{d\hat{y}}{dx} - \frac{dy}{dx} = b_1 + 2b_2x - (1 - y) = b_1 + 2b_2x + y - 1 = b_1 + 2b_2x + (2 + b_1x + b_2x^2) - 1 \\ &= b_1(1 + x) + b_2(2x + x^2) + 1 \end{aligned}$$

$$\frac{\partial e}{\partial b_1} = 1 + x$$

$$\frac{\partial e}{\partial b_2} = 2x + x^2$$

Using Eq. 8-42 of the textbook, the following results can be obtained:

$$\int_0^x [b_1(1 + x) + b_2(2x + x^2) + 1](1 + x) dx = 0$$

$$\int_0^x [b_1(1 + x) + b_2(2x + x^2) + 1](2x + x^2) dx = 0$$

Using $x = 1$ as the upper limit, above two equations yield the following two normal equations:

$$2.3333b_1 + 2.2500b_2 + 1.5000 = 0$$

$$2.2500b_1 + 2.5333b_2 + 1.3333 = 0$$

The solution of the normal equations is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.94293 \\ 0.31117 \end{bmatrix}$$

Thus, the quadratic approximation is given by

$$\hat{y} = 2 - 0.94293x + 0.31117x^2$$

For $x = 0.2$, $\hat{y} = 2 - 0.94293(0.2) + 0.31117(0.2)^2 = 1.82386$

The following table summarizes the estimated values of the function \hat{y} for the range of $x = 0$ to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	2	2	0.00000
0.2	1.823861	1.818731	0.28207
0.4	1.672615	1.670320	0.13741
0.6	1.546263	1.548812	0.16454
0.8	1.444805	1.449329	0.31216
1	1.368240	1.367879	0.02636

Problem 6:

Textbook: 8-35

$$\frac{dy}{dx} + y - 1 = 0 \quad \text{such that } y = 2 \text{ at } x = 0$$

$$\frac{dy}{dx} = 1 - y$$

$$\text{Quadratic Model: } \hat{y} = b_0 + b_1x + b_2x^2$$

Using the boundary condition

$$\hat{y} = 2 = b_0 + b_1(0) + b_2(0)^2$$

yields $b_0 = 2$. Thus, the quadratic model is

$$\hat{y} = 2 + b_1x + b_2x^2$$

and

$$\frac{d\hat{y}}{dx} = b_1 + 2b_2x$$

$$\begin{aligned} e &= \frac{d\hat{y}}{dx} - \frac{dy}{dx} = b_1 + 2b_2x - (1 - y) = b_1 + 2b_2x + y - 1 = b_1 + 2b_2x + (2 + b_1x + b_2x^2) - 1 \\ &= b_1(1 + x) + b_2(2x + x^2) + 1 \end{aligned}$$

$$\frac{d\hat{y}}{db_1} = x$$

$$\frac{d\hat{y}}{db_2} = x^2$$

Using Eq. 8-57 of the textbook, the following results can be obtained:

$$\int_0^x [b_1(1 + x) + b_2(2x + x^2) + 1](x) \, dx = 0$$

$$\int_0^x [b_1(1 + x) + b_2(2x + x^2) + 1](x^2) \, dx = 0$$

Using $x = 1$ as the upper limit, above two equations yield the following two normal equations:

$$0.83333b_1 + 0.91667b_2 + 0.50000 = 0$$

$$0.58333b_1 + 0.70000b_2 + 0.33333 = 0$$

The solution of the normal equations is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.91435 \\ 0.28577 \end{bmatrix}$$

Thus, the quadratic approximation is given by

$$\hat{y} = 2 - 0.91435x + 0.28577x^2$$

For $x = 0.2$, $\hat{y} = 2 - 0.91435(0.2) + 0.28577(0.2)^2 = 1.82856$

The following table summarizes the estimated values of the function \hat{y} for the range of $x = 0$ to 1 in an increment of 0.2. The table also compares the computed values with the true values.

x	Estimated $y(x)$	True $y(x)$	error(%)
0	2	2	0.00000
0.2	1.828561	1.818731	0.54049
0.4	1.679983	1.670320	0.57852
0.6	1.554267	1.548812	0.35224
0.8	1.451413	1.449329	0.14378
1	1.371420	1.367879	0.25884