# *INTRODUCTION TO ENGINEERING ECONOMICS*

• A. J. Clark School of Engineering •Department of Civil and Environmental Engineering

by

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ENCE 202
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Department of Civil and Environmental Engineering
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# *Types of Interest*

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■ Simple Interest

$$I = Pni$$

| | |
|---|---|
| P = principal | \$1,000 |
| i = interest rate | 0.12 |
| n = number of years or periods | 1 |
| I= interest | \$120.00 |

- Interest is due at the end of the time period. For fractions of a time period, multiply the interest by the fraction

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# Types of Interest



- Compound Interest: The interest of the interest.
  - A loan of $1,000 is made at an interest of 12% for 5 years. The interest is due at the end of each year with the principal is due at the end of the fifth year. The following table shows the resulting payment schedule:



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# Type of Interest

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| | | | | |
|---|---|---|---|---|
| P = principal | | $ , | | |
| i = interest rate | | . | | |
| n = number of years or periods | | | | |
| | | | | |
| Year | Amount at start of year | Interest at year end | Owed amount at year end | Payment |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ , . |

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# *Types of Interest*

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- Compound Interest (cont'd)
  - A loan of $1,000 is made at an interest of 12% for 5 years. The principal and interest are due at the end of the fifth year. The following table shows the resulting payment schedule:

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# *Types of Interest*

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i = interest rate

n = number of years or periods

| Year | Amount at start of year | Interest at year end | Owed amount at year end | Payment |
|---|---|---|---|---|
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ . |
| | $ , . | $ . | $ , . | $ , . |

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# Cash Flow

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- Cash flow over time: Upward arrow means positive flow, downward arrow means negative flow. There are two cash flows to each problem (borrower and lender flows).
- Net cash flow: The arithmetic sum of receipts (+) and disbursements (-) that occur at the same point in time.

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# Cash Flow

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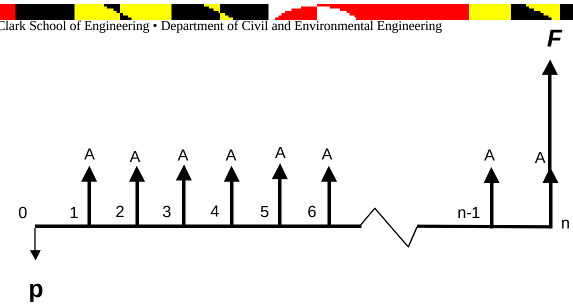


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# Notations

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*P* = a present single amount of money

*F* = a future single amount of money, after n periods of time

*A* = a series of n equal payments

*i* = the rate of interest per interest period (usually one year)

*n* = the number of periods of time (usually years)

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# Interest Formulas

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- Single Payment Compound-Amount Factor (SPCAF)

$$F = P(1+i)^n$$

OR

$$F = \left(\frac{F}{P}, i, n\right)$$

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# Interest Formulas

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- Example:
  - Let the principle $P$ = 1000, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - The future sum is therefore

$$F = P(1+i)^n = 1000(1+0.12)^4 = 1{,}573.5$$

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# Interest Formulas

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- Single Payment Present-Worth Factor (SPPWF)

$$P = \frac{P}{(1+i)^n}$$

OR

$$P = \left(\frac{P}{F}, i, n\right)$$

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# Interest Formulas

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- Example 1:
  - Let the principle $F$ = 1000, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - The present worth is therefore

$$P = \frac{F}{(1+i)^n} = \frac{1000}{(1+0.12)^4} = 635.5$$



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# Interest Formulas

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- Example 2:
  - Let the principle $F$ = 1,573.5, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - The present worth is therefore

$$P = \frac{F}{(1+i)^n} = \frac{1{,}573.5}{(1+0.12)^4} = 1000.0$$

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# Interest Formulas

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- Uniform (Equal payment) Series Compound-Amount Factor (USCAF)

$$F = A\left(\frac{(1+i)^n - 1}{i}\right)$$

OR

$$F = \left(\frac{F}{A}, i, n\right)$$

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# Interest Formulas

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- Example:
  - Let $A$ = 100, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - The future value is therefore

$$F = A\left(\frac{(1+i)^n - 1}{i}\right) = 100\left(\frac{(1+0.12)^4 - 1}{0.12}\right) = 477.9$$

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# Interest Formulas

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- Uniform (Equal payment) Series Sinking-Fund Factor (USSFF)

$$A = F\left(\frac{i}{(1+i)^n - 1}\right)$$

OR

$$A = \left(\frac{A}{F}, i, n\right)$$

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# Interest Formulas

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- Example:
  - Let the future value *F* = 1000, the interest rate *i* = 12%, and the number of periods *n* = 4 years.
  - Therefore

$$A = F\left(\frac{i}{(1+i)^n - 1}\right) = 1000\left(\frac{0.12}{(1+0.12)^4 - 1}\right) = 209.2$$

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# Interest Formulas



- Uniform (Equal payment) Series Capital-Recovery Factor (USCRF)

$$A = P\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right)$$

OR

$$A = \left(\frac{A}{P}, i, n\right)$$

**NOTE: This is the case of loans (mortgages)**





# Interest Formulas



- Example:
  - Let the present worth $P$ = 1000, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - Therefore

$$A = P\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right) = 1000\left(\frac{0.12(1+0.12)^4}{(1+0.12)^4 - 1}\right) = 329.2$$

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# Interest Formulas

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- Uniform (Equal payment) Series Present-Worth Factor (USPWF)

$$P = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

OR

$$P = \left(\frac{P}{A}, i, n\right)$$

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# Interest Formulas

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- Example:
  - Let $A$ = 100, the interest rate $i$ = 12%, and the number of periods $n$ = 4 years.
  - Therefore

$$P = A\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) = 100\left(\frac{(1+0.12)^4 - 1}{0.12(1+0.12)^4}\right) = 303.7$$

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# Interest Formulas



■ Uniform Gradient-Series Factor

– The gradient (*G*) is a value in the cash flow that starts with 0 at the end of year 1, *G* at the end of year 2, 2*G* at the end of year 3, and so on to (*n*-1)*G* at the end of year *n*

$$P = G\left(\frac{1}{i}\frac{n}{(1+i)^n - 1}\right)$$



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# Interest Formulas

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■ Example:

– Let *G* = 100, the interest rate *i* = 12%, and the number of periods *n* = 4 years.

– Therefore

$$A = G\left(\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right) = 100\left(\frac{1}{0.12} - \frac{4}{(1+0.12)^4 - 1}\right) = 135.9$$

# Discrete and Continuous Compounding



- Nominal Interest Rate
  - It is expressed in annual basis
  - Financial institutions refer to this rate as the *annual percentage rate* or *APR*
- Effective Interest Rate
  - It is an interest rate that is compounded using a time period less than a year
  - The nominal interest rate in this case is the effective rate times the number of compounding periods in a year
  - It is referred to as nominal rate compounded at the period less than a year



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# Discrete and Continuous Compounding

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- Example:
  - Effective rate is 1% per month
  - Therefore

    Nominal Rate = 1% (12) = 12% compounded monthly

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# Discrete and Continuous Compounding

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- Relationship Between the Effective Interest Rate for any given time interval and the Nominal Interest Rate per Year
- Define

$r$ =nominal interest rate per year
$i$ = effective interest rate in the time interval
$l$ = length of the time interval (in years)
$m$ = reciprocal of the length of the compounding period (in years)

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# Discrete and Continuous Compounding

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- The effective interest rate

$$i = \left(1 + \frac{r}{m}\right)^{lm} - 1$$

$r$ =nominal interest rate per year
$i$ = effective interest rate in the time interval
$l$ = length of the time interval (in years)
$m$ = reciprocal of the length of the compounding period (in years)

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# Discrete and Continuous Compounding



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- If the interest is compounded only once in the time interval, then $l$ ($m$) =1, and

$$i = \frac{r}{m}$$

*r* =nominal interest rate per year
*i* = effective interest rate in the time interval
*l* = length of the time interval (in years)
*m* = reciprocal of the length of the compounding period (in years)



# Discrete and Continuous Compounding

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- To find the applicable effective interest rate for any time interval, the following equation can be used:

$$i = \left(1 + \frac{r}{m}\right)^c - 1 \qquad c \geq 1$$

*r* =nominal interest rate per year
*i* = effective interest rate in the time interval
*c* = number of compounding periods in the time interval
*m* = reciprocal of the length of the compounding period (in years)

# Discrete and Continuous Compounding



■ Continuous Compounding

- The limiting case for the effective rate is when compounding is performed an infinite times in a year, that is continuously. Using *l* = 1, the following limit produces the continuously compounded interest rate ( $i_a$):

$$i_a = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m - 1 = e^r - 1$$





# Discrete and Continuous Compounding



■ Example 1

If the nominal rate of 12% is compounded monthly with time interval of one year, then c =12, and

$$i = \left(1 + \frac{012}{12}\right)^{12} - 1 = 0.1268 \text{ or } 12.68\% \text{ per year}$$

# *Discrete and Continuous Compounding*

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■ Example 2

If the nominal rate of 18% is compounded weekly with time interval of one year, then c =52, and

$$i = \left(1 + \frac{018}{52}\right)^{52} - 1 = 0.1968 \text{ or } 19.68\% \text{ per year}$$





# *Discrete and Continuous Compounding*

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■ Example 3

If the nominal rate of 14% is compounded monthly with time interval of six months, then c =6, and

$$i = \left(1 + \frac{014}{12}\right)^{6} - 1 = 0.0721 \text{ or } 7.21\% \text{ per six months}$$

# Discrete and Continuous Compounding

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- Example 4

If the nominal rate of 9% is compounded semiannually with time interval of two years, then

c =4, and

$$i = \left(1 + \frac{0.09}{2}\right)^4 - 1 = 0.1925 \text{ or } 19.25\% \text{ per two years}$$





# Discrete and Continuous Compounding

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- Example 5

The effective interest rates corresponding to a nominal annual interest rate of 18% compounded annually, semiannually, quarterly, monthly, weekly, daily, and continuously are shown in the following Table:

# Discrete and Continuous Compounding

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- Example 5 (cont'd)

| Compounding frequency Column 1 | Number of periods per year Column 2 | Effective interest rate per period Col. 3 = 18%/Col 2 | Effective annual interest rate $i = \left(1 + \frac{0.18}{Col.2}\right)^{Col.2} - 1$ |
|---|---|---|---|
| Annually | 1 | 18% | 18% |
| Semiannually | 2 | 9 | 18.81 |
| Quarterly | 4 | 4.5 | 19.2517 |
| Monthly | 12 | 1.5 | 19.5618 |
| Weekly | 52 | 0.3642 | 19.6843 |
| Daily | 365 | 0.0493 | 19.7412 |
| Continuously | $\infty$ | 0.00000 | 19.7217 = exp(0.18) - 1 |



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# Discrete and Continuous Compounding

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- Comparing Interest Rates
  - Since the effective interest rate represents the actual interest earned, this rate should be used to compare the benefits of various nominal rates of interest
  - For example, one might be confronted with the problem of determining whether it is more desirable to receive 16% compounded annually or 15% compounded monthly

# Discrete and Continuous Compounding



- Comparing Interest Rates
  - The effective rate of interest per year for 16% compounded annually is, of course, 16%
  - However, for 15% compounded monthly, the effective annual rate is

$$i_a = \left(1 + \frac{0.15}{12}\right)^{12} - 1 = 16.08\%$$

  - Thus, 15% compounded monthly yields an actual rate of interest that is higher than 16% compounded annually





# Use of Interest Tables



- To aid in calculations using the relationships between *P*, *F*, *A*, *n*, and *i*, you can also find values for *F/A*, *A/F*, *P/A*, and *A/P* in tables for typical values for *i* and *n*
- The following table gives these factors for *i* = 23 and

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| | $i$ = 20% | | | | | |
|---|---|---|---|---|---|---|
| $n$ | $P/F$ | $P/A$ | $A/F$ | $A/P$ | $F/P$ | $F/A$ |
| 1 | 0.8333333 | 0.8333333 | 1.0000000 | 1.2000000 | 1.200000 | 1.000000 |
| 2 | 0.6944444 | 1.5277780 | 0.4545455 | 0.6545455 | 1.440000 | 2.200000 |
| 3 | 0.5787037 | 2.1064810 | 0.2747253 | 0.4747253 | 1.728000 | 3.640000 |
| 4 | 0.4822531 | 2.5887350 | 0.1862891 | 0.3862891 | 2.073600 | 5.368000 |
| 5 | 0.4018776 | 2.9906120 | 0.1343797 | 0.3343797 | 2.488320 | 7.441600 |
| 6 | 0.3348980 | 3.3255100 | 0.1007057 | 0.3007057 | 2.985984 | 9.929920 |
| 7 | 0.2790816 | 3.6045920 | 0.0774239 | 0.2774239 | 3.583181 | 12.915900 |
| 8 | 0.2325680 | 3.8371600 | 0.0606094 | 0.2606094 | 4.299817 | 16.499080 |
| 9 | 0.1938067 | 4.0309670 | 0.0480795 | 0.2480795 | 5.159780 | 20.798900 |
| 10 | 0.1615056 | 4.1924720 | 0.0385228 | 0.2385228 | 6.191736 | 25.958680 |
| 11 | 0.1345880 | 4.3270600 | 0.0311038 | 0.2311038 | 7.430084 | 32.150420 |
| 12 | 0.1121567 | 4.4392170 | 0.0252650 | 0.2252650 | 8.916100 | 39.580500 |
| 13 | 0.0934639 | 4.5326810 | 0.0206200 | 0.2206200 | 10.699320 | 48.496600 |
| 14 | 0.0778866 | 4.6105670 | 0.0168931 | 0.2168931 | 12.839180 | 59.195920 |
| 15 | 0.0649055 | 4.6754730 | 0.0138821 | 0.2138821 | 15.407020 | 72.035110 |
| 16 | 0.0540879 | 4.7295610 | 0.0114361 | 0.2114361 | 18.488430 | 87.442130 |
| 17 | 0.0450732 | 4.7746340 | 0.0094401 | 0.2094401 | 22.186110 | 105.930600 |
| 18 | 0.0375610 | 4.8121950 | 0.0078054 | 0.2078054 | 26.623330 | 128.167000 |
| 19 | 0.0313009 | 4.8434960 | 0.0064625 | 0.2064625 | 31.948000 | 154.740000 |
| 20 | 0.0260841 | 4.8695800 | 0.0053565 | 0.2053565 | 38.337600 | 186.688000 |
| 21 | 0.0217367 | 4.8913160 | 0.0044439 | 0.2044439 | 46.005120 | 225.025600 |
| 22 | 0.0181139 | 4.9094300 | 0.0036896 | 0.2036896 | 55.206140 | 271.030700 |
| 23 | 0.0150949 | 4.9245250 | 0.0030653 | 0.2030653 | 66.247370 | 326.236900 |
| 24 | 0.0125791 | 4.9371040 | 0.0025479 | 0.2025479 | 79.496850 | 392.484200 |
| 25 | 0.0104826 | 4.9475870 | 0.0021187 | 0.2021187 | 95.396220 | 471.981100 |
| 26 | 0.0087355 | 4.9563230 | 0.0017625 | 0.2017625 | 114.475500 | 567.377300 |
| 27 | 0.0072796 | 4.9636020 | 0.0014666 | 0.2014666 | 137.370600 | 681.852800 |
| 28 | 0.0060663 | 4.9696680 | 0.0012207 | 0.2012207 | 164.844700 | 819.223300 |
| 29 | 0.0050553 | 4.9747240 | 0.0010162 | 0.2010162 | 197.813600 | 984.068000 |
| 30 | 0.0042127 | 4.9789360 | 8.461E-40 | 0.2008461 | 237.376300 | 1,181.882000 |
| 35 | 0.0016930 | 4.9915350 | 3.392E-40 | 0.2003392 | 590.668200 | 2,948.341000 |
| 40 | 6.804E-40 | 4.9965980 | 1.362E-40 | 0.2001362 | 1,469.772 | 7,343.858000 |
| 45 | 2.734E-40 | 4.9986330 | 5.470E-50 | 0.2000547 | 3,657.262 | 18,281.310000 |
| 50 | 1.099E-40 | 4.9994510 | 2.198E-50 | 0.2000220 | 9,100.438 | 45,497.190000 |
| 60 | 1.775E-50 | 5.9999110 | 3.529E-60 | 0.2000035 | 56,347.51 | 281,732.6 |
| 70 | 2.866E-60 | 4.9999860 | 5.732E-70 | 0.2000006 | 348,889.0 | 1,744,440.0 |
| 80 | 4.629E-70 | 4.9999980 | 9.258E-80 | 0.2000001 | 2,160,228.0 | 10,801,137.0 |
| 90 | 7.476E-80 | 5.0000000 | 1.495E-80 | 0.2000000 | 13,375,565 | 66,877,822 |
| 100 | 1.207E-80 | 5.0000000 | 2.145E-90 | 0.2000000 | 82,817,975 | 4.1409E8 |

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# *Use of Interest Tables*

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- Example
  - Let $A$ = 300, the interest rate $i$ = 20%, and the number of periods $n$ = 5 years.
  - Using the table, then

$$P = \left(\frac{P}{A}, i, n\right) = \left(\frac{P}{A}, 20, 5\right) = 300(2.9906) = 897.2$$

$$F = \left(\frac{F}{A}, i, n\right) = \left(\frac{F}{A}, 20, 5\right) = 300(7.4416) = 2{,}232.5$$

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