University of Maryland, College Park
Department of Civil and Environmental Engineering

Quiz 4 Solution, **closed book & notes, for 15 minutes**
November 12, 2001

ENCE 302
Probability and Statistics for Civil Engineers **Name:___________________**

Problem 1

The maximum impact pressure of ocean waves on coastal structures may be determined by

$$\rho_{\max} = 2.7\frac{\rho K V^2}{D}$$

Where ρ = density of water, *K* = length of hypothetical piston, *D* = thickness of air cushion, *V* = horizontal velocity of advancing wave. Suppose that the mean crest velocity *V* is 4.5 ft/sec with COV of 0.2. Note that ρ, *K*, and *D* are constants. If ρ = 1.96 slugs/cu ft, and the ratio *K*/*D* = 35, determine the mean and standard deviation of the peak impact pressure.

***** SOLUTION *****

$$\mathrm{E}(\rho_{\max}) = \mu_{\rho_{\max}} \approx g[\mathrm{E}(V)] = 2.7(1.96)(35)(4.5)^2 = 3750.7 \text{ psf}$$

and

$$\left.\frac{d\rho_{\max}}{dV}\right|_{V=4.5} = \frac{d}{dV}\left(2.7\frac{\rho K V^2}{D}\right) = 2(2.7)\frac{\rho K V}{D} = 2(2.7)(1.96)(35)(4.5) = 1{,}666.98$$

$$\therefore \mathrm{Var}(\rho_{\max}) \approx \left(\left.\frac{d\rho_{\max}}{dV}\right|_{V=4.5}\right)^2 \mathrm{Var}(V) = (1{,}666.98)^2(0.2\times 4.5)^2 = 2{,}250{,}846.1 \text{ psf}^2$$

$$\therefore \sigma_{\rho_{\max}} = \sqrt{2{,}250{,}846.1} = 1{,}500.3 \text{ psf}$$

Problem 1

The study duration and grade point average (GPA) of students graduating with B.S. degree from an engineering school were studied. With *X* defined as the number of years it takes to graduate and *Y* as the GPA, it was observed that *X* could be 4, 5, or 6 years and *Y* could be 2, 3, or 4. The following table shows the number of students for each combination of *X* and *Y*.

Y \ X	4	5	6
2	5	15	60
3	50	80	20
4	20	40	10

(a) Find the joint probability mass function (PMF) for *X* and *Y*.
(b) Determine the marginal PMF of *X* and the marginal PMF of *Y*.
(c) If only a GPA of 3 is under consideration (i.e., $Y = 3$), determine the conditional PMF of *X*.

***** SOLUTION *****

(a) Joint probability mass function (PMF) for *X* and *Y*:
Total # of students = 5+15+60+50+80+20+20+40+10 = 300 students

Y \ X	4	5	6
2	0.0167	0.0500	0.2000
3	0.1667	0.2667	0.0667
4	0.0667	0.1333	0.0333

(b) Marginal PMF of *X*:

$$P_{X_1}(x_1) = \sum_{\text{all } x_2} P_{X_1 X_2}(x_1, x_2) \qquad P_{X_2}(x_2) = \sum_{\text{all } x_1} P_{X_1 X_2}(x_1, x_2)$$

$$P_X(4) = 0.0167 + 0.1667 + 0.0667 = 0.25$$
$$P_X(5) = 0.0500 + 0.2667 + 0.1333 = 0.45$$
$$P_X(6) = 0.2000 + 0.0667 + 0.0333 = 0.30$$

Marginal PMF of Y:

$$P_Y(2) = 0.0167 + 0.0500 + 0.2000 = 0.267$$
$$P_Y(3) = 0.1667 + 0.2667 + 0.0667 = 0.500$$
$$P_Y(5) = 0.0667 + 0.1333 + 0.0333 = 0.233$$

(c) If only a GPA of 3 is under consideration (i.e., $Y = 3$), the conditional PMF of *X* will be:

$$P_{X|Y}(x_i \mid 3) = \frac{P_{X,Y}(x_i, 3)}{P_Y(3)}$$

Therefore,

$$P_{X|Y}(4 \mid 3) = \frac{0.1667}{0.5} = 0.3334$$

$$P_{X|Y}(5 \mid 3) = \frac{0.2667}{0.5} = 0.5334$$

$$P_{X|Y}(6 \mid 3) = \frac{0.0667}{0.5} = 0.1334$$

Formulas

Approximate Methods (Random Vector)

– First-order (approximate) Mean

$$E(Y)=\mu_Y = g[E(X_1), E(X_2), \ldots, E(X_n)]$$

– First-order (approximate) Variance

$$\mathrm{Var}(Y)=\sigma_Y^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{E(X_i)} \left.\frac{\partial g(\mathbf{X})}{\partial X_j}\right|_{E(X_j)} \mathrm{Cov}(X_i, X_j)$$

Approximate Methods (Random Vector)

– First-order (approximate) Variance

If the X_i's are uncorrelated (statistically independent), then

$$\mathrm{Var}(Y)=\sigma_Y^2 \approx \sum_{i=1}^{n}\left(\left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{E(X_i)}\right)^2 \mathrm{Var}(X_i)$$

$$\mathrm{P}(A \mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \qquad \mathrm{P}(B)\neq 0$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{X})^2$$

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right]$$

OR

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\overline{X}^2\right]$$

Marginal Distributions

The marginal mass function for X_2 that is not equal to zero is

$$P_{X_2}(x_2)=\sum_{\text{all } x_1} P_{X_1X_2}(x_1, x_2)$$

The marginal mass function for X_1 that is not equal to zero is

$$P_{X_1}(x_1)=\sum_{\text{all } x_2} P_{X_1X_2}(x_1, x_2)$$

Conditional Probability Mass Function

The conditional probability mass function for two random variables X_1 and X_2 is given by

$$P_{X_1|X_2}(x_1 \mid x_2)=\frac{P_{X_1X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

where $P_{X_1|X_2}(x_1 \mid x_2)$ results in the probability of $X_1 = x_1$ given that $X_2 = x_2$.

$P_{X_2}(x_2)=$ marginal mass function for X_2

Multiple Random Variables

– If the function $Y = g(X)$ is given by

$$Y = g(\mathbf{X}) = a_0 + a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

Then

$$E(Y)=a_0 + a_1E(X_1)+a_2E(X_2)+\ldots+a_nE(X_n)$$

and

$$\mathrm{Var}(Y)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}(X_i, X_j)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \rho_{X_iX_j}\sigma_{X_i}\sigma_{X_j}$$

If the random variables of $\mathbf{X}$ are uncorrelated, then

$$\mathrm{Var}(Y)=\sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$$