CHAPTER 7b

SIMULATION

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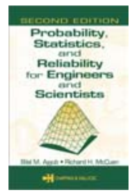

Probability and Statistics for Civil Engineers

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CHAPMAN HALL/CRC

Simulation Methods

- Simulation is the process conducting experiments on a model.
- A model is a representation for the real system or the real component for the purpose of studying the performance.
  - High cost.
  - Difficulty (impossibility).

Simulation Methods

- Monte Carlo techniques are techniques for testing engineering systems by imitating their real behavior.
- the accuracy of the simulation estimator increases as the simulation cycles increase.

Simulation Methods

- The performance function is defined as

$$Z = R - L = g(X_1, X_2, X_3, \ldots, X_n)$$

$X_1, X_2, X_3, \ldots, X_n$ = n random variables

If $Z > 0$ survival.
If $Z < 0$ failure.
If $Z = 0$ limit state.

Simulation Methods

- The reliability of each component in the system is the probability that the strength of the component exceeds the applied loadings on the same component.
- The probability of failure of the component is the probability that the strength of the component is less than the applied loadings on the component.

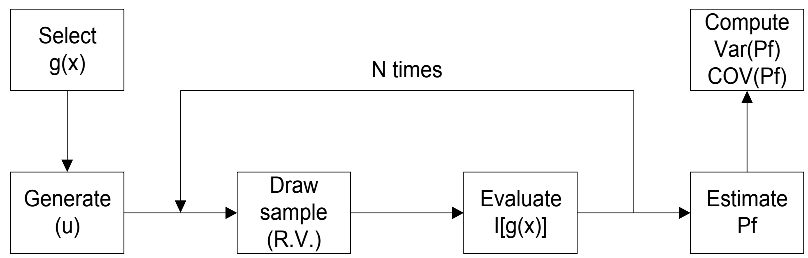
Simulation Methods

The Monte Carlo Simulation Methods:

- Direct Monte Carlo simulation
- Variance reduction techniques
  - Improve the simulation accuracy: reduce the variance of the estimated probability of failure,
  - Improve the simulation efficiency: reduce the number of simulation cycles

Simulation Methods

- Steps for simulations based variance reduction techniques:



Simulation Methods

- In the simulation techniques, compute:
  - The estimated probability of failure.
  - The variance of the estimated probability of failure.
  - The coefficient of variation of the estimated probability of failure.
  - The computational time (CPU).
  - The relative efficiency ratio $= \frac{\sigma_1^2 T_1}{\sigma_2^2 T_2}$

Simulation Methods

The VRT's are classified based on their common characteristics:

1. The importance sampling category: More samples are taken from the region of interest.
2. The correlated sampling category: Linear correlation among the randomly generated variables.
3. The conditional expectation category: Conditioning on one or more of the generated random variables.
4. The general techniques category: Individual characteristics.

Simulation Methods

Table 1. Classification of Variance Reduction Techniques.

Importance sampling category: • Importance sampling technique. • Adaptive sampling technique. • Stratified sampling technique. • Poststratified sampling technique. • Latin hypercube sampling technique. • Updated Latin hypercube sampling technique. • Spherical sampling technique. • Truncated sampling technique.
Correlated sampling category: • Antithetic Variate technique. • Common Random Numbers technique. • Control Variate technique. • Rotation Sampling technique.
Conditional expectation category: • Conditional expectation technique. • Generalized conditional expectation technique. • Adaptive hybrid conditional expectation technique.
General techniques category: • Response surface technique. • Adaptive response surface technique. • Russian roulette technique. • Russian roulette and splitting technique. • Jackknife technique.

VRT: Direct Monte Carlo Technique (DMC)

- Draw samples of the basic random variables based on their probabilistic characteristics and feeding them in the performance function.

$$Z = g(\underline{X}) = g(X_1, X_2, \ldots, X_n)$$

$$\overline{P}_f = \frac{N_f}{N}$$

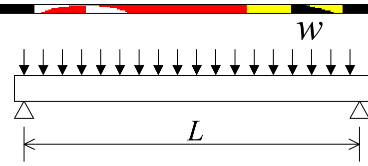
$$Var(\overline{P}_f) = \frac{(1 - \overline{P}_f)\overline{P}_f}{N}$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$$

VRT: Direct Monte Carlo Technique (DMC)

- Simulation steps for DMC:
  1. Select a performance function and identify its random variables and their probabilistic characteristics.
  2. Generate random numbers (u) and then the random variables values by using the inverse transformation method.
  3. Evaluate the performance function (limit state function), $g(\underline{X})$, add 1 to the failure counter,($I(.)$), if $g < 0$ and add 0 if $g \geq 0$.
  4. Repeat steps 2 to 3 N times.
  5. Determine the number of failures, N_f, based on the counter (I) value.
  6. Compute $\overline{P}_f = \frac{N_f}{N}$ and $Var(\overline{P}_f) = \frac{(1 - \overline{P}_f)\overline{P}_f}{N}$
  7. Compute $COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$

Example (DMC)



- Moment failure mode of a steel beam subjected to uniformly distribute loading.

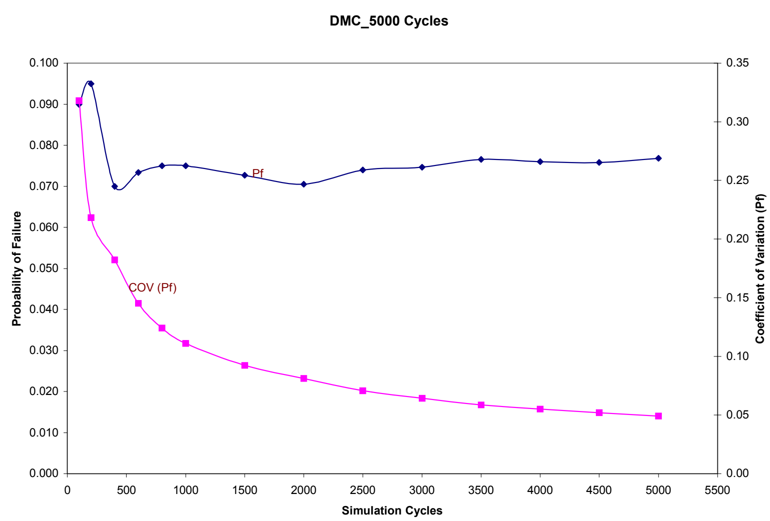
$$Z = F_y S - M$$

Where

Random Variable	Mean Value	COV	Distribution Type
Yield stress (F_y)	190 MPa	0.125	Normal
Section Modulus (S)	8.19×10^{-4} m^3	0.050	Normal
Load moment, (M)	1.13×10^5 N-m	0.200	Normal

F_y = material yield stress.
S = elastic section modulus.
M = moment effect due to applied loading.

Example (DMC)



VRT: Importance Sampling Technique (IS)

- The simulation samples are concentrated in the failure region.
- The random variables are generated according to selected probability distributions with mean values closer to the design point.

$$P_f = \int_{D_f} f_X(X)dX = \int_{D_f} \frac{f_X(X)}{h_X(X)} h_X(X)dX$$

$$\overline{P}_f = \frac{1}{N}\sum_{i=1}^{N} I[g(X) \le 0]\frac{f_X(X)}{h_X(X)}$$

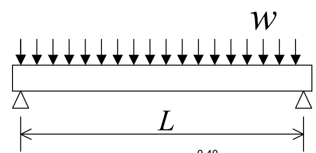
$$Var(\overline{P}_f) = \frac{\sum_{i=1}^{N}(P_{f_i} - \overline{P}_f)^2}{N(N-1)} \qquad COV\left(\overline{P}_f\right) = \frac{\sqrt{Var\left(\overline{P}_f\right)}}{\overline{P}_f}$$

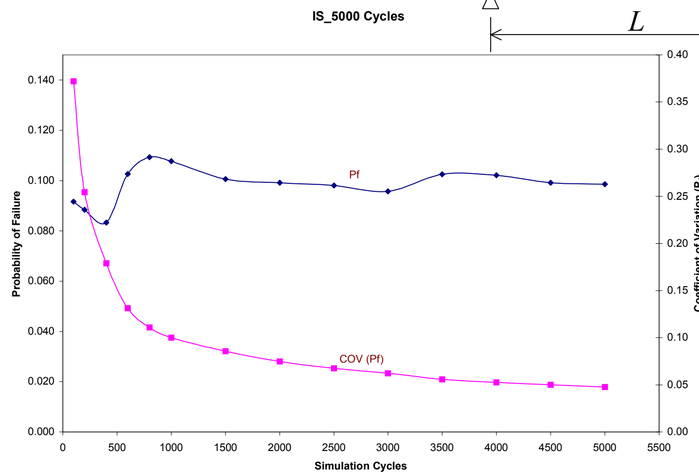
VRT: Importance Sampling Technique (IS)

- Simulation steps for IS:
  1. Select a performance function and identify its random variables and their probabilistic characteristics.
  2. Select the importance density function, $h_X(x)$, and define the original density function, $f_X(x)$.
  3. Generate random numbers (u) and then the random variables values by using the inverse transformation method.
  4. Evaluate the performance function (limit state function), $g(\underline{X})$, add 1 to the failure counter,($I(.)$), if $g < 0$ and add 0 if $g \ge 0$.
  5. Repeat steps 3 to 4 N times.
  6. Compute $\overline{P}_f = \frac{1}{N}\sum_{i=1}^{N} I[g(X_i) \le 0]\frac{f_X(X_i)}{h_X(X_i)}$ $\qquad Var(\overline{P}_f) = \frac{\sum_{i=1}^{N}(P_{f_i} - \overline{P}_f)^2}{N(N-1)}$
  7. Compute $COV\left(\overline{P}_f\right) = \frac{\sqrt{Var\left(\overline{P}_f\right)}}{\overline{P}_f}$

Example (IS)

$$Z = F_y S - M$$





VRT: Conditional Expectation Technique (CE)

- Randomly generating all the basic random variables except one variable.
- Select the control variable X_k with the highest variability.

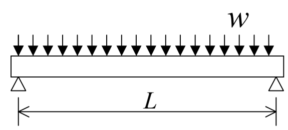
$$P_{f_i} = F_{X_k}\left[g_k\left(X_i : i = 1,2,\ldots,n; i \neq k\right)\right]$$

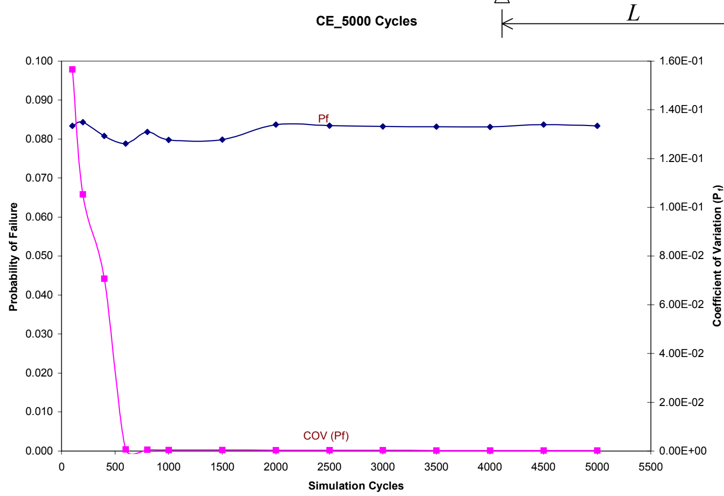
$$\overline{P}_f = \frac{\sum_{i=1}^{N} P_{f_i}}{N} \qquad Var(\overline{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \overline{P}_f)^2}{N(N-1)}$$

$$COV\left(\overline{P}_f\right) = \frac{\sqrt{Var\left(\overline{P}_f\right)}}{\overline{P}_f}$$

Example (CE)

$$Z = F_y S - M$$





VRT: Generalized Conditional Expectation Technique (GCE)

- The number of control variables are considered more than one.

$$P_{f_i} = 1 - F\left(\frac{\mu_{F_y} S_i - \mu_M}{\sqrt{S_i^2 \sigma_{F_y}^2 + \sigma_M^2}}\right)$$

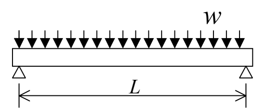
$$\overline{P}_f = \frac{\sum_{i=1}^{N} P_{f_i}}{N}$$

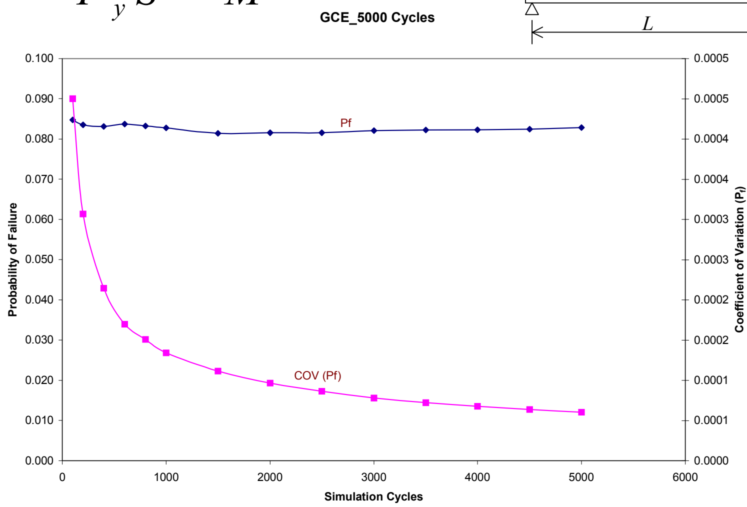
$$Var(\overline{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \overline{P}_f)^2}{N(N-1)}$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$$

Example (GCE)

$$Z = F_y S - M$$





VRT: Antithetic Variate Technique (AV)

- Negative correlation between different cycles of simulation is induced in order to decrease the variance of the estimated mean value.
- u and 1-u are used.

$$P_{f_i} = \frac{P_{f_i}^{(1)} + P_{f_i}^{(2)}}{2}$$

$$Var\left(\overline{P}_f\right) = \frac{1}{4N}\left[Var\left(P_f^{(1)}\right) + Var\left(P_f^{(2)}\right) + 2\,Cov\left(P_f^{(1)}, P_f^{(2)}\right)\right]$$

$$COV\left(\overline{P}_f\right) = \frac{\sqrt{Var\left(\overline{P}_f\right)}}{\overline{P}_f}$$

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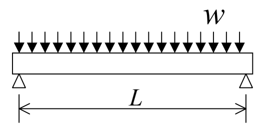
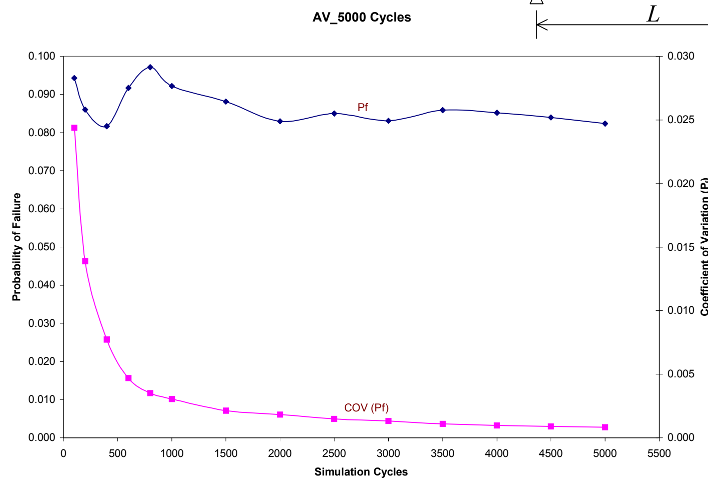
Example (AV)

$$Z = F_y S - M$$





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VRT: Stratified Sampling Technique (SS)

- The failure domain is divided into several disjoint subdomains. More samples are then taken from the most important subdomains.

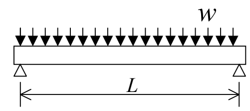
$$\overline{P}_f = \sum_{i=1}^{5} P_i \frac{1}{N_i} \sum_{i=1}^{N_i} I[g(X) < 0]$$

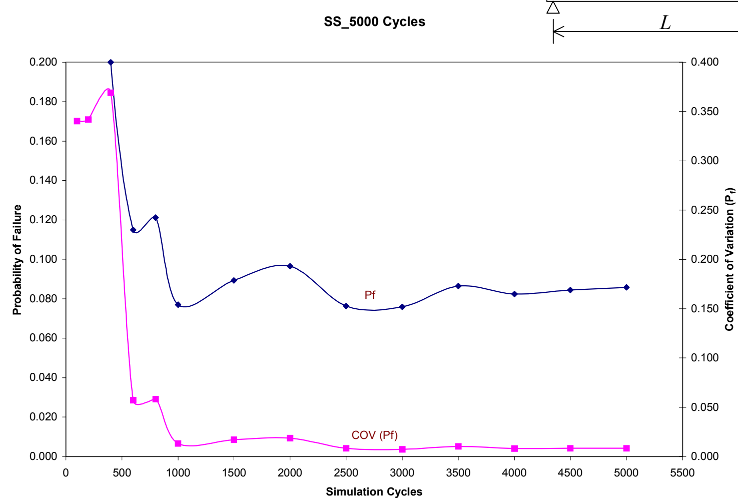
$$Var(\overline{P}_f) = \sum_{i=1}^{5} \frac{P_i^2 \sigma_i^2}{N_i}$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$$

Example (SS)

$$Z = F_y S - M$$





VRT: Control Variate Technique (CV)

- Takes advantage of correlation between certain variables.
- Another random variable with known mean is selected to adjust the P_f .
- Generate an initial run to estimate the adjustment constant, $a = \frac{Cov(g(X), g_o(X))}{Var(g_o(X))}$

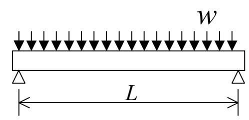
$$\overline{P}_f = \frac{1}{N}\sum_{i=1}^{N}\left[I[g(x_i) < 0] - a(I(g_o(x_i) < 0))\right] + a\mu_c$$

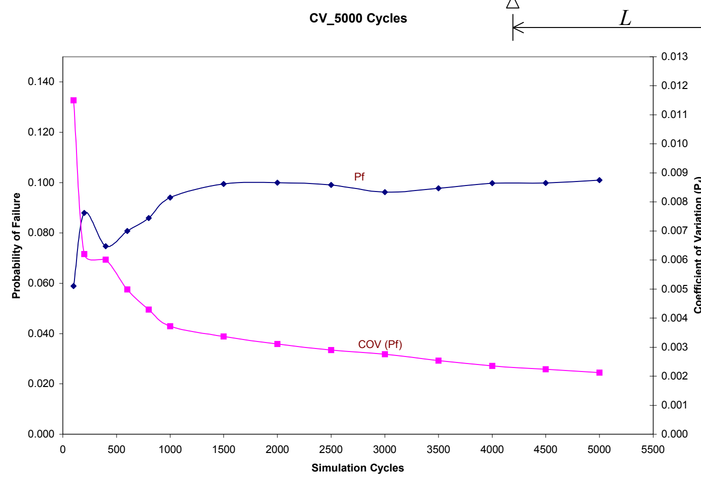
$$Var\left[\overline{P}_f\right] = Var(g(X)) + a^2 Var(g_o(X)) - 2aCov(g(X), g_o(X))$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$$

Example (CV)

$$Z = F_y S - M$$





VRT: Russian Roulette Technique (RR)

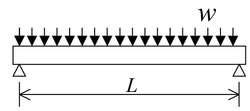
- Some simulation cycles are killed (ceased to exist) by chance with a certain probability.
- The survival probability is determined and the the survival weight is then adjusted as $\overline{w} = \dfrac{1}{N(P_{survival})}$
- J_n is the Russian roulette counter for survived simulation cycles.

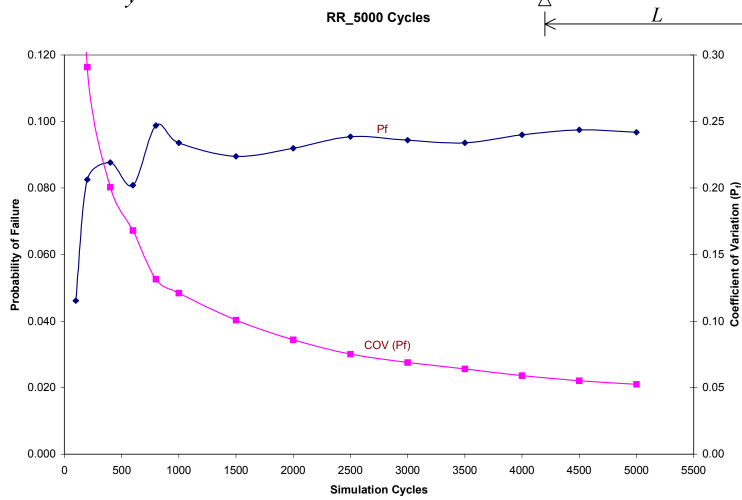
$$\overline{P}_f = \sum_{i=1}^{N} I[g(X_i) \le 0].J_n.\overline{w} \qquad Var(\overline{P}_f) = \frac{\sum_{i=1}^{N} (P_{f_i} - \overline{P}_f)^2}{N(N-1)}$$

$$COV(\overline{P}_f) = \frac{\sqrt{Var(\overline{P}_f)}}{\overline{P}_f}$$

Example (RR)

$$Z = F_y S - M$$





VRT: Russian Roulette & Splitting Technique (RR&S)

- The killed samples in the Russian Roulette are compensated in the splitting process in this technique.
- Pre-determine the number of splits, k.
- Wight the samples as $(1/k)$.

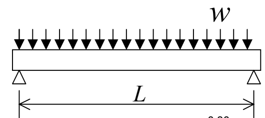
$$\overline{P}_f = \frac{1}{N}\sum_{i=1}^{N} I[g(X_i) \le 0].J_n$$

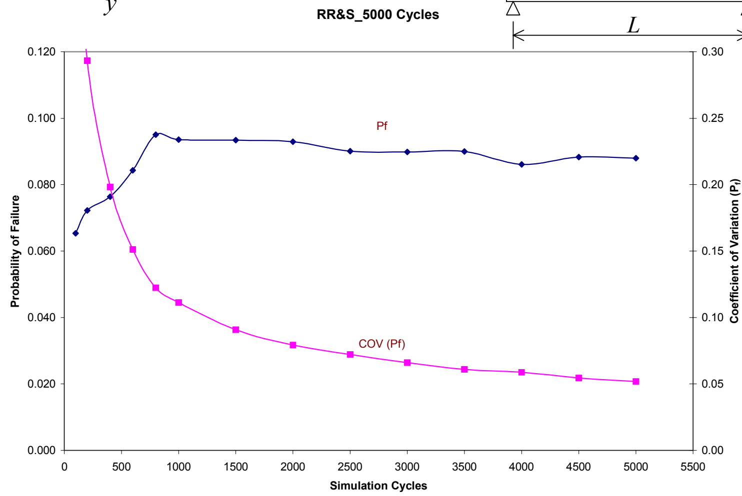
where J_n = Russian roulette counter

$$Var(\overline{P}_f) = \frac{\sum_{i=1}^{N}(P_{f_i} - \overline{P}_f)^2}{N(N-1)} \qquad COV\left(\overline{P}_f\right) = \frac{\sqrt{Var\left(\overline{P}_f\right)}}{\overline{P}_f}$$

Example (RR&S)

$$Z = F_y S - M$$

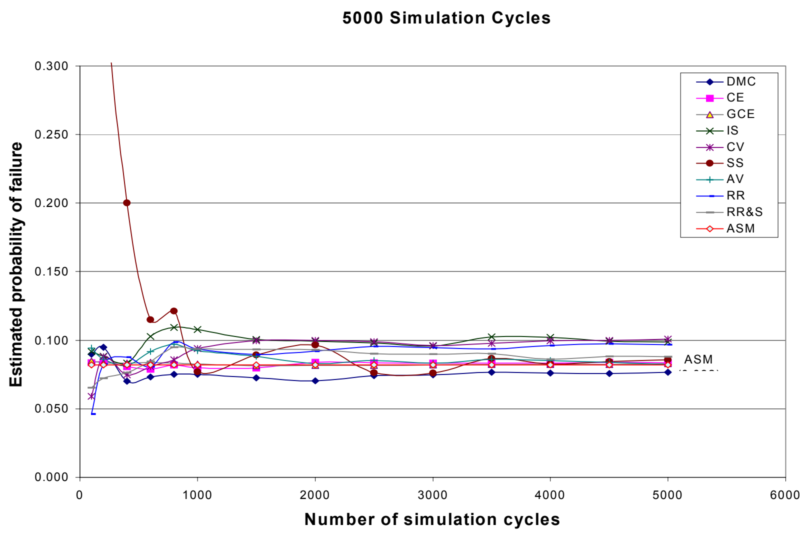




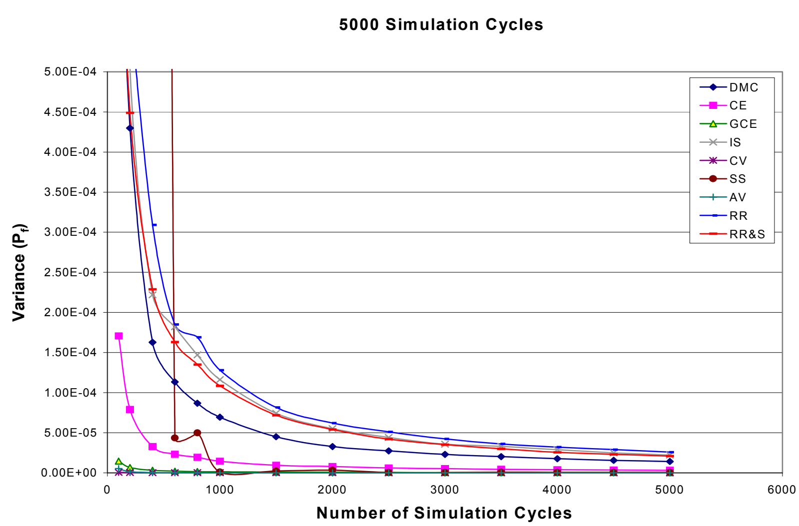
Example (Results for All VRT's)

VRT	1000			3000			5000					
	Pf	COV(Pf)	Time	Pf	COV(Pf)	Time	Pf	COV(Pf)	S.D.	VAR	Time	*Relative Efficiency Ratio*
DMC	0.0750	0.1111	22.7	0.0747	0.0643	61.0	0.0768	4.90E-02	3.77E-03	1.42E-05	98.0	**1**
CE	0.0798	0.0005	21.1	0.0832	0.0003	63.4	0.0834	2.09E-04	1.75E-05	3.05E-10	105.5	**43160**
GCE	0.0828	0.0001	12.8	0.0821	0.0001	38.8	0.0828	6.04E-05	5.00E-06	2.50E-11	66.7	**832689**
IS	0.1077	0.1001	32.0	0.0957	0.0625	94.0	0.0986	4.77E-02	4.70E-03	2.21E-05	155.9	**0.4**
CV	0.0941	0.0037	28.8	0.0962	0.0028	102.7	0.1010	2.12E-03	2.14E-04	4.60E-08	198.7	**152**
SS	0.0770	0.0133	15.9	0.0760	0.0074	48.0	0.0858	8.31E-03	7.13E-04	5.08E-07	80.1	**34**
AV	0.0922	0.0030	40.2	0.0831	0.0013	137.3	0.0824	8.42E-04	6.94E-05	4.82E-09	247.6	**1165**
RR	0.0936	0.0012	19.1	0.0944	0.0007	56.4	0.0968	5.25E-02	5.08E-03	2.58E-05	93.7	**1**
RR&S	0.0935	0.0011	23.2	0.0898	0.0007	69.3	0.0880	5.19E-02	4.57E-03	2.08E-05	114.7	**1**
ASM	0.0820			0.0820			0.0820					

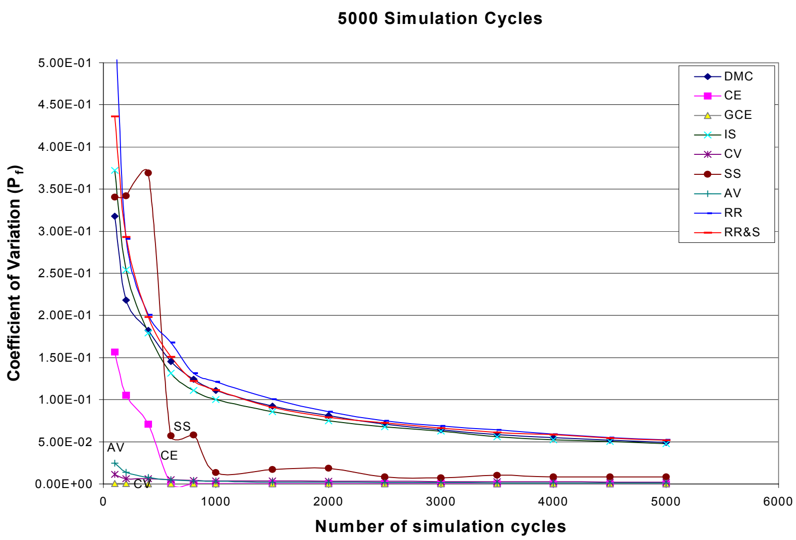
Example (Results for All VRT's)



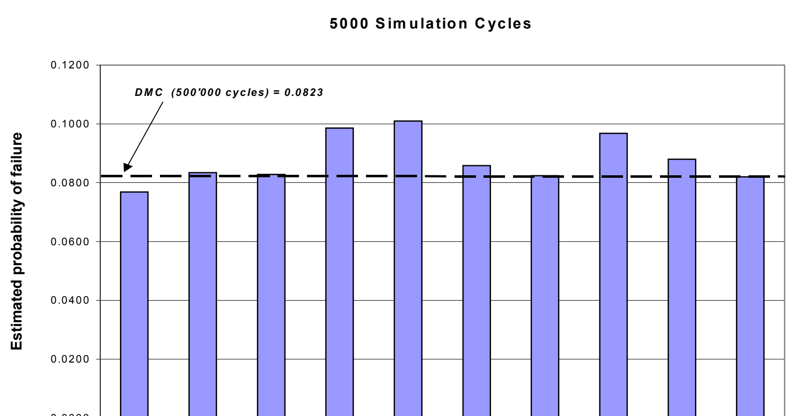
Example (Results for All VRT's)



Example (Results for All VRT's)



Example (Results for All VRT's)



	DMC	CE	GCE	IS	CV	SS	AV	RR	RR&S	ASM
Series1	0.0768	0.0834	0.0828	0.0986	0.1010	0.0858	0.0824	0.0968	0.0880	0.0820

Techniques

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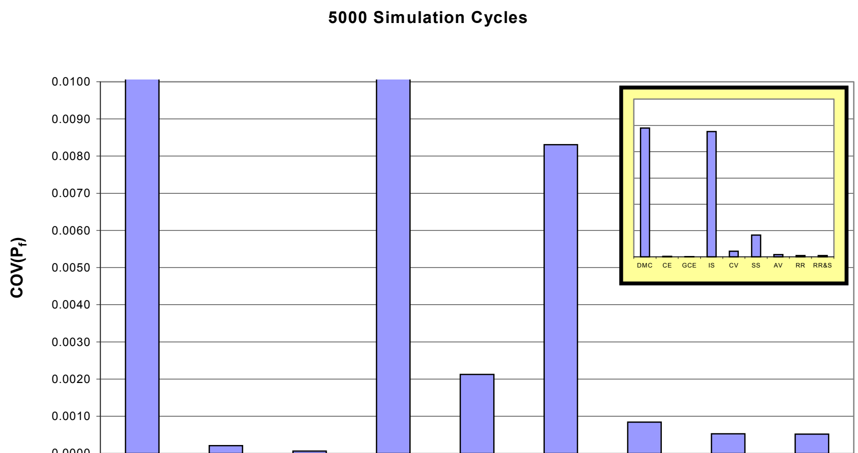
Example (Results for All VRT’s)



	DMC	CE	GCE	IS	CV	SS	AV	RR	RR&S
Series1	0.0490	0.0002	0.0001	0.0477	0.0021	0.0083	0.0008	0.0005	0.0005

Techiques

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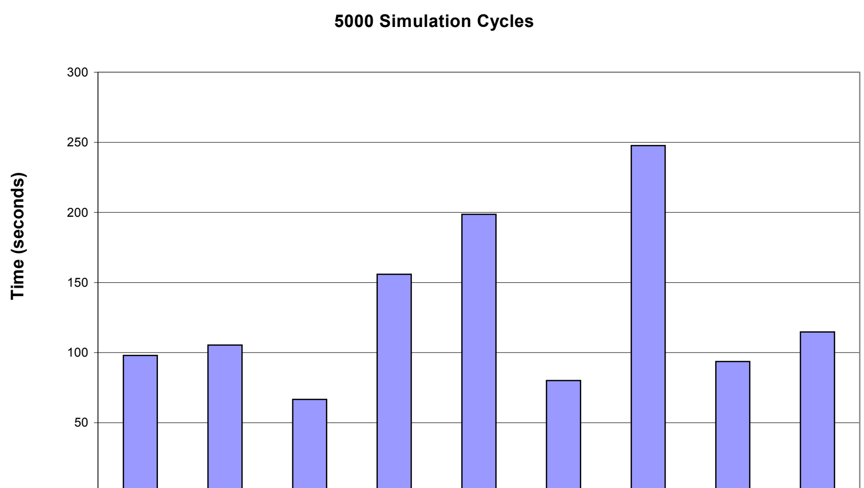
Example (Results for All VRT’s)



	DMC	CE	GCE	IS	CV	SS	AV	RR	RR&S
Series1	98.0	105.5	66.7	155.9	198.7	80.1	247.6	93.7	114.7

Techniques

Example (Results for All VRT's)