CHAPTER

Probability, Statistics, and Reliability
for Engineers and Scientists

Second Edition

MULTIPLE RANDOM VARIABLES

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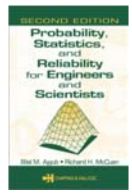

Probability and Statistics for Civil Engineers
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CHAPMAN
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Functions of Random Variables

- Approximate vs. Analytical Methods
  - Need for approximate methods
    - In general there are few closed-form engineering solutions for the distribution types of the dependent variable $Y = g(\boldsymbol{X})$.
    - Mathematical expectation of this function provides solutions for simple cases, i.e., linear random variables.
    - For cases that involve a more complex $g(\boldsymbol{X})$, or a mixture of distribution types, exact solutions can be a difficult task due to analytical complexity, especially when $g(\boldsymbol{X})$ is nonlinear.

Functions of Random Variables

- Analytical Methods
  - A more general case, in which the functional relationship between the dependent variable *Y* and the basic random variable *X* is not linear, is considered.

$$Y = g(X)$$

  - If *Y* is monotonically increasing function of *X*, then $\mathrm{P}(Y \leq y) = \mathrm{P}(X \leq x)$

Functions of Random Variables

- Analytical Methods

Or

$$F_Y(y) = F_X(x) = F_X\left[g^{-1}(y)\right]$$

If both sides are differentiated with respect to *y*, the PDF of *Y* can be obtained as

$$f_Y(y) = \frac{dF_X(x)}{dy} = f_X\left[g^{-1}(y)\right]\frac{dg^{-1}(y)}{dy}$$

Functions of Random Variables

- Analytical Methods

  If *Y* decreases with *X*, then $dg^{-1}(y)/dy$ can be negative. Since the PDF of a random variable cannot be negative, its absolute value is of interest. Therefore, to account for this, the PDF of *Y* is written as

$$f_Y(y) = f_X\left[g^{-1}(y)\right]\left|\frac{dg^{-1}(y)}{dy}\right|$$

Functions of Random Variables

- Analytical Methods

  In many cases, the inverse function $g^{-1}(y)$ may have *n* values x_i, and if the $f_X(x_i)$ are nonzero positive numbers, the PDF of *Y* is expressed as

$$f_Y(y) = \sum_{i=1}^{n} f_X\left[g_i^{-1}(y)\right]\left|\frac{dg_i^{-1}(y)}{dy}\right|$$

Functions of Random Variables

- Example: Analytical Method

Assume the following quadratic relationship between Y and X:

$$Y = cX^2$$

where c = constant.

Also. Assume that X is a normal random variable with mean μ_X and standard deviation σ_X.

The PDF of X is given by

$$f_X(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \quad \text{for } -\infty < x < +\infty$$

Functions of Random Variables

- Example (cont'd): Analytical Method

$$Y = cX^2$$

if the above equation is inverted, the two roots of X can be shown to be

$$x = g^{-1}(y) = \pm\sqrt{\frac{y}{c}}$$

and

$$\frac{dx}{dy} = \frac{dg^{-1}(y)}{dy} = \pm\frac{1}{2}\frac{1}{\sqrt{cy}}$$

Functions of Random Variables

- Example (cont'd): Analytical Method

For a normal distribution, the PDF of *Y* becomes

$$f_Y(y) = \frac{1}{2\sqrt{cy}}\left[f_X\left(+\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)\right]$$

or

$$f_Y(y) = \frac{1}{\sigma_X\sqrt{2\pi}\left(2\sqrt{cy}\right)}\left[e^{-\frac{1}{2}\left(\frac{+\sqrt{\frac{y}{c}}-u_X}{\sigma_X}\right)^2} + e^{-\frac{1}{2}\left(\frac{-\sqrt{\frac{y}{c}}-u_X}{\sigma_X}\right)^2}\right]$$

Functions of Random Variables

- **Approximate Methods**
  – Taylor Series Expansion

A Taylor series is commonly used in engineering analysis to approximate functions that do not have closed form solution. The Taylor series is given by

$$f(x_0+h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

where

x_0 = base value or starting value

x = the point at which the value of the function is needed

$h = x - x_0$ = distance between x_0 and x (step size)

$n!$ = factorial of $n = n(n-1)(n-2)\ldots1$

$f^{(n)}$ = indicates the n^{th} derivative of the function $f(x)$

R_{n+1} = the remainder of Taylor series expansion

Functions of Random Variables

- Approximate Methods
  - Taylor Series Expansion
    - First-order approximation

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0)$$

    - Second-order approximation

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!} f^{(2)}(x_0)$$

    - Third-order approximation

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!} f^{(2)}(x_0) + \frac{h^3}{3!} f^{(3)}(x_0)$$

Functions of Random Variables

- Approximate Methods
  - The Taylor series expansion can be used to approximate the mean and variance of a function of random variables $Y = g(\boldsymbol{X})$
  - Two cases to be considered:
    1. Single random variable X
    2. Multiple random variables, a random vector $\boldsymbol{X}$

Functions of Random Variables

- Approximate Methods
  – Single Random Variable *X*

    The Taylor series expansion of a function $Y = g(X)$ about the mean of X (E(X)) is given by

$$Y = g(X) = g(\mu_X + h) = g(\mu_X) + h\frac{dg(X)}{dX}\bigg|_{\mu_X} + \frac{1}{2!}h^2\frac{d^2g(X)}{dX^2}\bigg|_{\mu_X} + \ldots + \frac{1}{k!}h^k\frac{d^kg(X)}{dX^k}\bigg|_{\mu_X}$$

or

$$Y = g(\mu_X) + [X - \mu_X]\frac{dg(X)}{dX}\bigg|_{\mu_X} + \frac{1}{2}[X - \mu_X]^2\frac{d^2g(X)}{dX^2}\bigg|_{\mu_X} + \ldots + \frac{1}{k!}[X - \mu_X]^k\frac{d^kg(X)}{dX^k}\bigg|_{\mu_X}$$

Functions of Random Variables

- Approximate Methods
  – Single Random Variable *X*

$$g(X) = g[\mathrm{E}(X)] + [X - \mathrm{E}(X)]\frac{dg(X)}{dX}\bigg|_{\mathrm{E}(X)} + \frac{1}{2}[X - \mathrm{E}(X)]^2\frac{d^2g(X)}{dX^2}\bigg|_{\mathrm{E}(X)}$$
$$+ \ldots + \frac{1}{k!}[X - \mathrm{E}(X)]^k\frac{d^kg(X)}{dX^k}\bigg|_{\mathrm{E}(X)}$$

If the series is truncated at the second term, then

$$g(X) = g[\mathrm{E}(X)] + [X - \mathrm{E}(X)]\frac{dg(X)}{dX}\bigg|_{\mathrm{E}(X)}$$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

$$g(X) = g[E(X)] + [X - E(X)] \left.\frac{dg(X)}{dX}\right|_{E(X)}$$

Taking the expectation of both sides, and noting that

$$E[X - E(X)] = E(X) - E[E(X)] = E(X) - E(X) = 0$$

Hence,

$$E(Y) = E[g(X)] \approx g[E(X)] \approx g(\mu_X)$$

Functions of Random Variables

- Approximate Methods
  - Single Random Variable *X*

$$g(X) = g[E(X)] + [X - E(X)] \left.\frac{dg(X)}{dX}\right|_{E(X)}$$

Taking the variances of both sides, and noting that

$$\text{Var}[g[E(X)]] = \text{Var}[g(\mu_X)] = 0$$

Hence,

$$E(Y) = \text{Var}\left[[X - E(X)] \left.\frac{dg(X)}{dX}\right|_{E(X)}\right] = \left[\left.\frac{dg(X)}{dX}\right|_{E(X)}\right]^2 \text{Var}(X)$$

Functions of Random Variables

- Approximate Methods (single RV)
  - First-order (approximate) Mean

$$\mathrm{E}(Y) = \mu_Y = g[\mathrm{E}(X)]$$

  - First-order (approximate) Variance

$$\mathrm{Var}(Y) = \sigma_Y^2 = \left(\left. \frac{dg(X)}{dX} \right|_{\mathrm{E}(X)} \right)^2 \mathrm{Var}(X)$$

Functions of Random Variables

- Example: Pressure of Ocean Waves

The maximum impact pressure of ocean waves on coastal structures may be determined by

$$\rho_{\max} = 2.7 \frac{\rho K V^2}{D}$$

Where ρ = density of water, *K* = length of hypothetical piston, *D* = thickness of air cushion, *V* = horizontal velocity of advancing wave. Suppose that the mean crest velocity *V* is 4.5 ft/sec with *COV* of 0.2. ρ, *K*, and *D* are constants. If ρ = 1.96 slugs/cu ft, and the ratio *K*/*D* = 35, determine the mean and standard deviation of the peak impact pressure.

Functions of Random Variables

- Example (cont'd): Pressure of Ocean Waves

$$E(\rho_{max}) = \mu_{\rho_{max}} \approx g[E(V)] = 2.7(1.96)(35)(4.5)^2 = 3750.7 \text{ psf}$$

and

$$\left.\frac{d\rho_{max}}{dV}\right|_{V=4.5} = \frac{d}{dV}\left(2.7\frac{\rho K V^2}{D}\right) = 2(2.7)\frac{\rho K V}{D} = 2(2.7)(1.96)(35)(4.5) = 1{,}666.98$$

$$\therefore \text{Var}(\rho_{max}) \approx \left(\left.\frac{d\rho_{max}}{dV}\right|_{V=4.5}\right)^2 \text{Var}(V) = (1{,}666.98)^2(0.2\times 4.5)^2 = 2{,}250{,}846.1 \text{ psf}^2$$

$$\therefore \sigma_{\rho_{max}} = \sqrt{2{,}250{,}846.1} = 1{,}500.3 \text{ psf}$$

Functions of Random Variables

- Approximate Methods (Random Vector)
  - First-order (approximate) Mean

$$E(Y) = \mu_Y = g[E(X_1), E(X_2), \ldots, E(X_n)]$$

  - First-order (approximate) Variance

$$\text{Var}(Y) = \sigma_Y^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left.\frac{\partial g(\mathbf{X})}{\partial X_i}\right|_{E(X_i)}\left.\frac{\partial g(\mathbf{X})}{\partial X_j}\right|_{E(X_i)}\text{Cov}(X_i, X_j)$$

Functions of Random Variables

- Approximate Methods (Random Vector)
  - First-order (approximate) Variance

    If the X_i's are uncorrelated (statistically independent), then

$$\mathrm{Var}(Y) = \sigma_Y^2 \approx \sum_{i=1}^{n} \left(\left. \frac{\partial g(\boldsymbol{X})}{\partial X_i} \right|_{\mathrm{E}(X_i)} \right)^2 \mathrm{Var}(X_i)$$

Functions of Random Variables

- <u>Example 1:</u>

  Assume that the random variable *Y* can be represented by the following relationship:

$$Y = X_1 X_2^2 X_3^{1/3}$$

  where X_1, X_2, and X_3 are statistically independent random variables with mean values of 1.0, 1.5, and 0.8, respectively, and corresponding standard deviations of 0.1, 0.2, and 0.15, respectively. Find the first-order mean and standard deviation of *Y*.

Functions of Random Variables

- Example 1 (cont'd):

$$Y = X_1 X_2^2 X_3^{1/3}$$

$$\mathrm{E}(Y) = \mu_Y = g[\mathrm{E}(X_1), \mathrm{E}(X_2), \mathrm{E}(X_3)]$$
$$= (1.0)(1.5)^2(0.8)^{1/3} = 2.0887$$

$$\left.\frac{\partial Y}{\partial X_1}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_1}\right|_{\mu_{X_i}} = \left(X_2^2 X_3^{1/3}\right)\Big|_{\mu_{X_i}} = \mu_{X_2}^2 \mu_{X_3}^{1/3}$$

$$\left.\frac{\partial Y}{\partial X_2}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_2}\right|_{\mu_{X_i}} = \left(2X_1 X_2 X_3^{1/3}\right)\Big|_{\mu_{X_i}} = 2\mu_{X_1}\mu_{X_2}\mu_{X_3}^{1/3}$$

$$\left.\frac{\partial Y}{\partial X_3}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(X_1 X_2^2 X_3^{1/3}\right)}{\partial X_2}\right|_{\mu_{X_i}} = \left.\left(\frac{1}{3}\frac{X_1 X_2^2}{X_3^{2/3}}\right)\right|_{\mu_{X_i}} = \frac{\mu_{X_1}\mu_2^2}{3\mu_{X_3}^{2/3}}$$

Functions of Random Variables

- Example 1 (cont'd):

$$\left(\left.\frac{\partial Y}{\partial X_1}\right|_{\mu_{X_i}}\right)^2 = \left(\mu_{X_2}^2 \mu_{X_3}^{1/3}\right)^2 = \left[(1.5)^2(0.8)^{1/3}\right]^2 = 4.3627$$

$$\left(\left.\frac{\partial Y}{\partial X_2}\right|_{\mu_{X_i}}\right)^2 = \left(2\mu_{X_1}\mu_{X_2}\mu_{X_3}^{1/3}\right)^2 = \left[2(1)(1.5)(0.8)^{1/3}\right]^2 = 7.7560$$

$$\left(\left.\frac{\partial Y}{\partial X_3}\right|_{\mu_{X_i}}\right)^2 = \left(\frac{\mu_{X_1}\mu_2^2}{3\mu_{X_3}^{2/3}}\right)^2 = \left[\frac{(1)(1.5)^2}{3(0.8)^{2/3}}\right]^2 = 0.7574$$

Functions of Random Variables

■ Example 1 (cont'd):

$$\text{Var}(X_1) = \sigma_{X_1}^2 = (0.1)^2 = 0.01$$
$$\text{Var}(X_2) = \sigma_{X_2}^2 = (0.2)^2 = 0.04$$
$$\text{Var}(X_3) = \sigma_{X_3}^2 = (0.15)^2 = 0.0225$$

$$\text{Var}(Y) = \sigma_Y^2 \approx \sum_{i=1}^{3} \left(\left. \frac{\partial g(\boldsymbol{X})}{\partial X_i} \right|_{\text{E}(X_i)} \right)^2 \text{Var}(X_i)$$

$$\approx \left(\left. \frac{\partial g(Y)}{\partial X_1} \right|_{\text{E}(X_i)} \right)^2 \text{Var}(X_1) + \left(\left. \frac{\partial g(Y)}{\partial X_2} \right|_{\text{E}(X_i)} \right)^2 \text{Var}(X_2) + \left(\left. \frac{\partial g(Y)}{\partial X_3} \right|_{\text{E}(X_i)} \right)^2 \text{Var}(X_3)$$

$$\approx (4.3627)(0.01) + 7.7560(0.04) + 0.7574(0.0225) = 0.3709$$

$$\therefore \sigma_Y = \sqrt{0.3709} = \underline{0.609}$$

Functions of Random Variables

■ Example 2:

The stress F in a beam subjected to an external bending moment M is

$$F = \frac{My}{I}$$

where y is the distance from the neutral axis of the cross section of the beam to the point where the stress is calculated, and I = is the centriodal moment of inertia of the cross section. Assume that M and I are random variables with means μ_M and μ_I, respectively, and variances σ_M and σ_I, respectively.

Functions of Random Variables

- Example 2(cont'd):

Determine the mean and variance of F based on first-order approximation.

$$\mu_F = \frac{\mu_M y}{\mu_I}$$

$$\left.\frac{\partial F}{\partial M}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(\frac{My}{I}\right)}{\partial M}\right|_{\mu_{X_i}} = \left.\left(\frac{y}{I}\right)\right|_{\mu_{X_i}} = \frac{y}{\mu_I}$$

$$\left.\frac{\partial F}{\partial I}\right|_{\mu_{X_i}} = \left.\frac{\partial\left(\frac{My}{I}\right)}{\partial I}\right|_{\mu_{X_i}} = \left.\left(-\frac{My}{I^2}\right)\right|_{\mu_{X_i}} = -\frac{\mu_M y}{\mu_I^2}$$

Functions of Random Variables

- Example 2(cont'd):

$$\mathrm{Var}(F) = \sigma_F^2 \approx \sum_{i=1}^{2}\left(\left.\frac{\partial g(\boldsymbol{X})}{\partial X_i}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(X_i)$$

$$\approx \left(\left.\frac{\partial g(F)}{\partial M}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(M) + \left(\left.\frac{\partial g(F)}{\partial I}\right|_{\mathrm{E}(X_i)}\right)^2 \mathrm{Var}(I)$$

$$\approx \left(\frac{y}{\mu_I}\right)^2 \sigma_M^2 + \left(\frac{\mu_M y}{\mu_I^2}\right)^2 \sigma_I^2$$

Multivariable Simulation

- Simulation can be used to study the probabilistic characteristics of a function of random variables.
- It can provides information about the distributions of random variables that is beyond the ability of theory.
- Theoretical relationships are often based on restrictive assumptions, such as normal distributions, that may not be valid for a given problem.

Multivariable Simulation

■ Stress at Extreme Fibers of a Beam

- The stress at the extreme fibers of steel beam is given by

$$\sigma = \frac{Mc}{I}$$

- To estimate the mean and standard deviation of σ, we can use the first-order approximation as discussed previously, regardless of the distribution types of the basic random variables *c* and *I*.

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - Simulation can also be used to study the probabilistic characteristics of σ, such as the mean and standard deviation.
  - However, the distribution types of the basic random variables *c* and *I* are required.

Multivariable Simulation

- Stress at Extreme Fibers of a Beam

Random Variable	Mean	Standard Deviation	Distribution Type
c	10	0.5	Normal
M	3000	900	Lognormal
I	1000	80	Normal

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - First-order Approximate mean and standard deviation of *M*

$$\mu_\sigma = \frac{\overline{Mc}}{\bar{I}} = \frac{3000(10)}{1000} = 30$$

$$\left.\frac{\partial\sigma}{\partial M}\right|_{\bar{X}_i} = \left.\frac{c}{I}\right|_{\bar{X}_i} = \frac{10}{1000} = 0.01, \quad \left(\left.\frac{\partial\sigma}{\partial M}\right|_{\bar{X}_i}\right)^2 = 0.0001$$

$$\left.\frac{\partial\sigma}{\partial c}\right|_{\bar{X}_i} = \left.\frac{M}{I}\right|_{\bar{X}_i} = \frac{3000}{1000} = 3, \quad \left(\left.\frac{\partial\sigma}{\partial c}\right|_{\bar{X}_i}\right)^2 = 9$$

$$\left.\frac{\partial\sigma}{\partial I}\right|_{\bar{X}_i} = -\left.\frac{Mc}{I^2}\right|_{\bar{X}_i} = -\frac{3000(10)}{(1000)^2} = -0.03, \quad \left(\left.\frac{\partial\sigma}{\partial M}\right|_{\bar{X}_i}\right)^2 = 0.0009$$

$$\text{Var}(\sigma) = (0.0001)(900)^2 + 9(0.5)^2 + 0.0009(80)^2 = 89.01$$

$$\text{Standard Deviation } (\sigma) = \sqrt{89.01} = 9.43$$

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - Simulation result for the mean and standard deviation of *M*

For 1000 simulation cycles,

Mean of $\sigma = 30.15$

Standard Deviation of $\sigma = 9.67$

# Cycles	u1	u2	u3	*M*	*c*	*I*	σ
1	0.902062	0.735778	0.290168	4200.533	10.31519	955.7684	45.33453
2	0.94779	0.350819	0.607922	4628.371	9.808445	1021.913	44.42368
3	0.458328	0.615189	0.984024	2786.548	10.14643	1171.601	24.13239
4	0.450338	0.312857	0.78368	2770.104	9.756117	1062.775	25.42915
5	0.978812	0.734305	0.822165	5214.236	10.31294	1073.892	50.07405
	.	.	.	.	.	.	.
	.	.		.	.	.	.
1000	0.708552	0.253279	0.665872	3376.147	9.667897	1034.283	31.55831

Multivariable Simulation

- Stress at Extreme Fibers of a Beam
  - Comparison Between Approximate Method and Simulation

$\sigma = \frac{Mc}{I}$	Approximation	Simulation
Mean of σ	30.00	30.15
Standard Deviation of σ	9.43	9.67