Probability, Statistics, and Reliability for Engineers and Scientists

Second Edition

MULTIPLE RANDOM VARIABLES

• A. J. Clark School of Engineering •Department of Civil and Environmental Engineering

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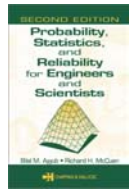

Probability and Statistics for Civil Engineers

Department of Civil and Environmental Engineering

University of Maryland, College Park

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Conditional Moments, Covariance, and Correlation Coefficient

- Computational Procedures for Moments
  - Discrete Random Variable:
    - The k[th] moment about the origin is given by

$$M_k' = \sum_{\text{all } x} x_1^k x_2^k \ldots x_n^k P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

  - Continuous Random Variable:
    - The k[th] moment about the origin is given by

$$M_k' = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} x_1^k x_2^k \ldots x_n^k f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n$$

Conditional Moments, Covariance, and Correlation Coefficient

- The previous moments are considered as a special case of *mathematical expectation*.
- The mathematical expectation of an arbitrary function $g(\boldsymbol{x})$, which is a function of the random vector $\boldsymbol{X}$, is defined in the following viewgraph for discrete and continuous cases.

Conditional Moments, Covariance, and Correlation Coefficient

- Mathematical Expectation
  - Discrete Random Variable:
    - The mathematical expectation is given by

$$E[g(\boldsymbol{X})] = \sum_{\text{all } \boldsymbol{x}} g(\boldsymbol{x}) P_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

  - Continuous Random Variable:
    - The mathematical expectation is given by

$$E[g(\boldsymbol{X})] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(\boldsymbol{x}) f_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n$$

Conditional Moments, Covariance, and Correlation Coefficient

- For simplicity, the presentation of the materials in the remaining part of this section is limited to *two* random variables.
- For the two-dimensional case, X_1 and X_2, the conditional mean for X_1 given that X_2 takes value x_2 denoted $\mu_{X1|X2}$, is defined in terms the conditional mass and density functions for the discrete and continuous random variables.

Conditional Moments, Covariance, and Correlation Coefficient

- Conditional Mean
  - Discrete Random Variable

$$\mu_{X_1|X_2} = \mathrm{E}(X_1 \mid X_2) = \sum_{\text{all } x_1} x_1 P_{X_1|X_2}(x_1 \mid x_2)$$

  - Continuous Random Variable

$$\mu_{X_1|X_2} = \mathrm{E}(X_1 \mid X_2) = \int_{-\infty}^{+\infty} x_1 f_{X_1|X_2}(x_1 \mid x_2)\, dx_1$$

Conditional Moments, Covariance, and Correlation Coefficient

- Conditional Mean

For statistically uncorrelated random variables X_1 and X_2, the conditional mean is given by

$$\mu_{X_1|X_2} = \mathrm{E}(X_1 \mid X_2) = E(X_1)$$

$$\mu_{X_2|X_1} = \mathrm{E}(X_2 \mid X_1) = E(X_2)$$

Also, it can be shown that

$$\mathrm{E}_{X_2}\left(\mu_{X_2|X_1}\right) = E(X_1)$$

Conditional Moments, Covariance, and Correlation Coefficient

- Conditional Variance
  - Discrete Random Variables

$$\mathrm{Var}(X_1 \mid X_2) = \sum_{\text{all } x_1} \left(x_1 - \mu_{X_1|X_2}\right)^2 P_{X_1|X_2}(x_1 \mid x_2)$$

  - Continuous Random Variables

$$\mathrm{Var}(X_1 \mid X_2) = \int_{-\infty}^{+\infty} \left(x_1 - \mu_{X_1|X_2}\right)^2 f_{X_1|X_2}(x_1 \mid x_2)\, dx_1$$

Conditional Moments, Covariance, and Correlation Coefficient

- The variance of a random variable X_1 can also be computed using conditional variance as follows:

$$\mathrm{Var}(X_1) = \mathrm{E}_{X_2}[\mathrm{Var}(X_1 \mid X_2)] + \mathrm{Var}_{X_2}[\mathrm{E}(X_1 \mid X_2)]$$

Conditional Moments, Covariance, and Correlation Coefficient

- Covariance of Two Random Variables

The covariance (Cov) of two random variables X_1 and X_2 is defined in terms of mathematical expectation as

$$\mathrm{Cov}(X_1, X_2) = \mathrm{E}[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})]$$

It is common to use the following notations for the covariance of X_1 and X_2:

$\sigma_{X_1 X_2}$, σ_{12}, or $\mathrm{Cov}(X_1, X_2)$

Conditional Moments, Covariance, and Correlation Coefficient

■ Covariance of Two Random Variables

- It can be shown that the Cov can also be determined using the following equation:

$$\text{Cov}(X_1, X_2) = \text{E}(X_1 X_2) - \mu_{X_1}\mu_{X_2}$$

where

$$\text{E}(X_1 X_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2)\, dx_1 dx_2$$

Conditional Moments, Covariance, and Correlation Coefficient

■ Covariance of Two Random Variables

– If X_1 and X_2 are statistically uncorrelated random variables, then

$$\text{Cov}(X_1, X_2) = 0$$

and

$$\text{E}(X_1 X_2) = \mu_{X_1}\mu_{X_2}$$

Conditional Moments, Covariance, and Correlation Coefficient

■ Correlation Coefficient

The correlation coefficient of two random variables X_1 and X_2 is defined as a normalized covariance with respect to the standard deviations of X_1 and X_2 and is given by

$$\rho_{X_1X_2} = \frac{\text{Cov}(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}$$

The correlation coefficient ranges between -1 and +1,

$$-1 \leq \rho_{X_1X_2} \leq +1$$

Conditional Moments, Covariance, and Correlation Coefficient

■ Correlation Coefficient

- If the correlation coefficient is zero, then the two random variables are said to be ***uncorrelated***.
- In order for ρ_{X1X2} to be zero, the Cov(X_1 X_2) must be zero.
- Therefore X_1 and X_2 are statistically uncorrelated.
- However, the converse of this finding is not true.
- The correlation coefficient can be viewed as a measure of the degree of linear association between X_1 and X_2.

Conditional Moments, Covariance, and Correlation Coefficient

- Example: Two Discrete RV's

Given the following joint density function of random variables X and Y and assume n = 2:

$$f_{XY}(x,y)=\begin{cases}\dfrac{n+1}{n-1} & \text{for } 0\le x\le 1 \text{ and } x^n\le y\le x^{\frac{1}{n}}\\ 0 & \text{otherwise}\end{cases}$$

(a) Find the marginal density functions of X and Y.

(b) Determine the mean or expected values of X and Y.

(c) The covariance and correlation coefficient of X and Y

Conditional Moments, Covariance, and Correlation Coefficient

- Example (cont'd): Two Discrete RV's

a) Marginal density functions

$$f_X(x)=\int_{x^n}^{x^{\frac{1}{n}}} f_{XY}(x,y)\,dy=\int_{x^n}^{x^{\frac{1}{n}}}\frac{n+1}{n-1}\,dy=\frac{n+1}{n-1}\left(x^{\frac{1}{n}}-x^n\right)$$

$$=\frac{2+1}{2-1}\left(x^{\frac{1}{2}}-x^2\right)=3\left(x^{\frac{1}{2}}-x^2\right)\qquad \text{for } 0\le x\le 1$$

$$f_Y(y)=\int_{y^n}^{y^{\frac{1}{n}}} f_{XY}(x,y)\,dx=\int_{y^n}^{y^{\frac{1}{n}}}\frac{n+1}{n-1}\,dx=\frac{n+1}{n-1}\left(y^{\frac{1}{n}}-y^n\right)$$

$$=\frac{2+1}{2-1}\left(y^{\frac{1}{2}}-y^2\right)=3\left(y^{\frac{1}{2}}-y^2\right)\qquad \text{for } 0\le y\le 1$$

Conditional Moments, Covariance, and Correlation Coefficient

- Example (cont'd): Two Discrete RV's
  - b) Expected values of *X* and *Y*

Since $f_X(x) = f_Y(y) \quad \Rightarrow \quad \mathrm{E}(X) = \mathrm{E}(Y)$

$$\therefore\ \mathrm{E}(X) = \mathrm{E}(Y) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(\frac{n+1}{n-1}\right)\left(x^{\frac{1}{n}} - x^n\right)dx$$

$$= \int_0^1 \left(\frac{n+1}{n-1}\right)\left(x^{\frac{n+1}{n}} - x^{n+1}\right)dx = \left(\frac{n+1}{n-1}\right)\left[\frac{x^{\frac{2n+1}{n}}}{\frac{2n+1}{n}} - \frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \left(\frac{n+1}{n-1}\right)\left[\frac{n}{2n+1} - \frac{1}{n+2}\right]$$

For $n = 2$, $\mathrm{E}(X) = \mathrm{E}(Y) = 0.45$

Conditional Moments, Covariance, and Correlation Coefficient

- Example (cont'd): Two Discrete RV's
  - c) Covariance and correlation of *X* and *Y*

Since $f_X(x) = f_Y(y) \quad \Rightarrow \quad \mathrm{E}(X^2) = \mathrm{E}(Y^2)$ or $\sigma_X^2 = \sigma_Y^2$

$$\therefore\ \mathrm{E}(X^2) = \mathrm{E}(Y^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(\frac{n+1}{n-1}\right)\left(x^{\frac{1}{n}} - x^n\right)dx$$

$$= \int_0^1 \left(\frac{n+1}{n-1}\right)\left(x^{\frac{1+2n}{n}} - x^{n+2}\right)dx = \left(\frac{n+1}{n-1}\right)\left[\frac{x^{\frac{1+3n}{n}}}{\frac{1+3n}{n}} - \frac{x^{n+3}}{n+3}\right]_0^1$$

$$= \left(\frac{n+1}{n-1}\right)\left[\frac{n}{1+3n} - \frac{1}{n+3}\right]$$

If $n = 2$, then $\sigma_X^2 = \sigma_Y^2 = 0.2571$

Conditional Moments, Covariance, and Correlation Coefficient

- <u>Example (cont'd):</u> Two Discrete RV's
  - c) Covariance and correlation (cont'd)

The variances of X and Y are computed as follows:

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 = 0.2571 - (0.45)^2 = 0.0546$$

The expected value of the product XY is

$$\mathrm{E}(XY) = \int_0^1 \int_{x^n}^{x^{\frac{1}{n}}} \frac{n+2}{n-1}\, xy\, dx dy = 0.25$$

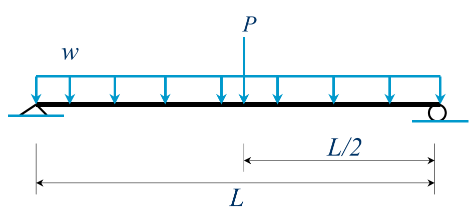
Therefore,

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 0.25 - 0.45(0.45) = 0.0475$$

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}} = \frac{0.0475}{\sqrt{0.0546}\sqrt{0.0546}} = 0.870$$

Functions of Random Variables

- Many engineering problems deal with a dependent variables that is a function of one or more independent variables



$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Functions of Random Variables

- Three cases to be considered:
  - Probability distributions for dependent random variables,
  - Mathematical expectations, and
  - Approximate methods

Functions of Random Variables

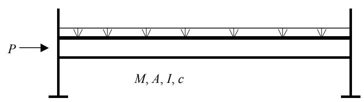
- A random variable *X* is defined as a mapping from a sample space of an engineering system or experiment to the real line of numbers.
- If *Y* is defined to be a dependent variable in terms of a function

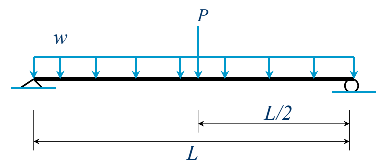
$$Y = g(X)$$

then *Y* is also a random variable

Functions of Random Variables

- Examples





$$\sigma = \frac{P}{A} + \frac{Mc}{I}$$

$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Functions of Random Variables

- Single Random Variable
  - The stress (*Y*) in a beam is a function of an applied load (*X*). If the load is random, the stress is also random

$$Y = g(X)$$

  - Linear Case

$$Y = g(X) = aX + b$$

where a and b are real numbers

$$\mathrm{E}(Y) = a\mathrm{E}(X) + b \qquad \mathrm{Var} = a^2 \mathrm{Var}(X)$$

Functions of Random Variables

- Multiple Random Variables
  - The stress (Y) in a beam is a function of an applied load, material properties, and geometry: $Y = g(X_1, X_2, ..., X_n)$
  - Linear Case

$$Y = g(\mathbf{X}) = a_0 + a_1 X_1 + a_2 X_2 + ... + a_n X_n$$

where a and b are real numbers

$$E(Y) = a_0 + a_1 E(X_1) + a_2 E(X_2) + ... + a_n E(X_n)$$

$$\text{Var}(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \text{Cov}(X_i, X_j)$$

Functions of Random Variables

- Multiple Random Variables
  - It should be noted that

$$\text{Cov}(X_i, X_i) = \text{Var}(X_i) = \sigma_X^2$$

  - The variance of Y can be also obtained from

$$\text{Var}(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{X_i X_j} \sigma_{X_i} \sigma_{X_j}$$

  - If the random variables of the vector $\boldsymbol{X}$ are statistically uncorrelated, then

$$\text{Var}(Y) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i)$$

Functions of Random Variables

- Example: Mean and Variance of a Linear Function

Assume X and Y are uncorrelated.

$Z = 2X + 5Y + 10$

$\mu_X = 3$ and $\mu_Y = 5$

$\sigma_X = 1$ and $\sigma_Y = 2$

Therefore,

$$\mu_Z = 2\mu_X + 5\mu_Y + 10 = 2(3) + 5(5) + 10 = 41$$

$$\sigma_Z^2 = \sum_{i=1}^{2} a_i^2 \operatorname{Var}(X_i) = 2^2(1)^2 + 5^2(2)^2 = 104$$

and

$$\sigma_Z = \sqrt{104} = 10.2 \qquad \operatorname{COV}(Z) = \frac{10.2}{41} = 0.25$$

Functions of Random Variables

- Mathematical Expectation

Mathematical expectation for $Y = g(X)$

– Discrete Case

$$\mathrm{E}[g(X)] = \sum_{i=1}^{n} g(x_i) P_X(x_i)$$

– Continuous Case

$$\mathrm{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

Functions of Random Variables

- Variance

  The variance for $Y = g(X)$

  – Discrete Case

$$\mathrm{Var}(Y) = \mathrm{Var}[g(X)] = \sum_{i=1}^{n} \left(g(x_i) - \mathrm{E}[g(x_i)]\right)^2 P_X(x_i)$$

  – Continuous Case

$$\mathrm{Var}(Y) = \mathrm{Var}[g(X)] = \int_{-\infty}^{+\infty} \left(g(x_i) - \mathrm{E}[g(x_i)]\right)^2 f_X(x_i)\,dx$$

Functions of Random Variables

- Special Case

  – If the function $Y = g(X) = a X + b$, then

$$\mathrm{E}(Y) = a\mathrm{E}(X) + b$$

$$\mathrm{Var}(Y) = a^2 \mathrm{Var}(X)$$

Where a and b are real numbers.

Functions of Random Variables

- Multiple Random Variables
  - If the function $Y = g(X)$ is given by

$$Y = g(\boldsymbol{X}) = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$$

Then

$$\mathrm{E}(Y) = a_0 + a_1 \mathrm{E}(X_1) + a_2 \mathrm{E}(X_2) + \ldots + a_n \mathrm{E}(X_n)$$

and

$$\mathrm{Var}(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \mathrm{Cov}(X_i, X_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{X_i X_j} \sigma_{X_i} \sigma_{X_j}$$

If the random variables of $\boldsymbol{X}$ are uncorrelated, then

$$\mathrm{Var}(Y) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$$

Functions of Random Variables

- Multiple Random Variables
  - If the function $Y = g(X)$ is given by

$$Y = g(\boldsymbol{X}) = X_1 X_2 X_3 \ldots X_n$$

Then

$$\mathrm{E}(Y) = \mathrm{E}(X_1)\mathrm{E}(X_2)\mathrm{E}(X_2)\ldots\mathrm{E}(X_n)$$

and

$$\mathrm{Var}(Y) = \mathrm{E}(X_1^2)\mathrm{E}(X_2^2)\ldots\mathrm{E}(X_n^2) - [\mathrm{E}(X_1)\mathrm{E}(X_1)\ldots\mathrm{E}(X_n)]^2$$

Functions of Random Variables

- <u>Example:</u> Cost of Precast Concrete

  The total cost C to manufacture a concrete panel in a precast plant is

  $$C = 1.5X + 2Y$$

  where X is the cost of materials, Y is the cost of labor. If the costs X and Y are assumed to be uncorrelated with means of $100/panel and $250/panel, respectively, and with standard deviations of $10/panel and $50/panel, respectively, compute the mean,variance, standard deviation, and COV of the total cost.

Functions of Random Variables

- <u>Example (cont'd):</u> Cost of Precast Concrete

$$\mu_C = 1.5\mu_X + 2\mu_Y$$
$$= 1.5(100) + 2(250)$$
$$= \$650/\text{panel}$$
$$\sigma_C^2 = 1.5^2\sigma_X^2 + 2^2\sigma_Y^2$$
$$= 1.5^2(10)^2 + 2^2(50)^2 = 10{,}225(\$/\text{panel})^2$$
$$\sigma_C = \sqrt{10{,}225} = \$101.12/\text{panel}$$
$$COV = \frac{101.12}{650} = 0.1556$$