Probability, Statistics, and Reliability for Engineers and Scientists

Second Edition

CHAPTER

MULTIPLE RANDOM VARIABLES

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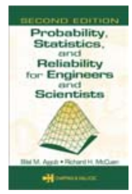

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Probability and Statistics for Civil Engineers

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Introduction

- In engineering, it is common to deal with two or more random variables simultaneously in solving problems.
- If the load applied to a structure is considered to be a random variable, then the structural response will also be a random variable.

Introduction

- The load and the response can be modeled separately as random variables; however, it is more prudent to model the uncertainty jointly.
- More information can be extracted from the joint distributions.
- Thus, it is necessary to extend the discussion to multiple random variables.

Introduction

- In general, multiple random variables are encountered in the following two forms:
  1. Joint occurrences of multiple random variables that can be correlated or uncorrelated
  2. Random variables that are known in terms of their functional relationship with other basic random variables

Joint Random Variables and Their Probability Distributions

- The outcomes, E_1, E_2,…, E_n, that constitute a sample space *S* are mapped to an *n*-dimensional (*n*-D) space of real numbers.
- The functions that establish such a transformation to the *n*-D space are called multiple random variables (or random vectors).

Joint Random Variables and Their Probability Distributions

- Multiple random variables are classified into two types:
  - Discrete random variables
  - Continuous random variables
- A distinction is made between these two types because the computations of probabilities depend on their type.

Probability for Discrete Random Vectors

- Joint Probability Mass Function (JPMF)

The joint probability mass function for a discrete multiple random variable or random vector ***X*** = (X_1, X_2,..,X_n) is given by

$$P_X(x) = \mathrm{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

Note that

$$0 \le \mathrm{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) \le 1$$

Probability for Discrete Random Vectors

- Joint Cumulative Mass Function (JCMF)

The joint cumulative mass function for a discrete random variable or random vector ***X*** = (X_1, X_2,..,X_n) is given by

$$F_X(x) = \mathrm{P}(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \sum_{\mathrm{all}(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)} P_X(x_1, x_2, \ldots, x_3)$$

Probability for Discrete Random Vectors

- Properties of JCMF
  1. $F_X(\text{all } x \to \infty) = 0$
  2. $F_X(x_1, x_2, \ldots, x_i \to -\infty, \ldots, x_n) = 0$, for any $i = 1, 2, \ldots, n$
  3. $F_X(x_1, x_2, \ldots, x_i \to -\infty, \ldots, x_k \to -\infty, \ldots, x_n) = 0$, for any values of $x_i, \ldots, x_k$
  4. $F_X(x_1, x_2, \ldots, x_i \to +\infty, \ldots, x_n) = F_{Xj}(x_j : j = 1, 2, \ldots, n$ and $j \neq i)$, called the marginal distribution of all the random variables except X_i
  5. $F_X(x_1, x_2, \ldots, x_i \to +\infty, \ldots, x_k \to +\infty, \ldots, x_n) = F_{Xj}(x_j : j = 1, 2, \ldots,$ n and $j \neq i$ to $k)$, called the marginal distribution of all the random variables except X_i to X_k
  6. $F_X(\text{all } x \to +\infty) = 1$
  7. $F_X(x)$ is a nonnegative and nondecreasing function of x

Probability for Discrete Random Vectors

- Properties of JCMF
  - The first, second, and third properties define the limiting behavior of $F_X(\boldsymbol{x})$; as one or more of the random variables approach $-\infty$, $F_X(\boldsymbol{x})$ approaches zero.
  - The fourth and fifth properties define the possible marginal distributions as one or more of the random variables approaches $+\infty$.
  - The sixth property is based on the probability axiom.
  - The seventh property is based on the cumulative nature of $F_X(\boldsymbol{x})$.

Probability for Two Discrete Random Variables

- For simplicity, the presentation of the materials in the remaining part of this section is limited to *two* random variables.
- The presented concepts can be generalized to *n* random variables

Probability for Two Discrete Random Variables

- Conditional Probability Mass Function

The conditional probability mass function for two random variables X_1 and X_2 is given by

$$P_{X_1|X_2}(x_1 \mid x_2) = \frac{P_{X_1X_2}(x_1, x_2)}{P_{X_2}(x_2)}$$

where $P_{X_1|X_2}(x_1 \mid x_2)$ results in the probability of $X_1 = x_1$ given that $X_2 = x_2$.

$P_{X_2}(x_2) =$ marginal mass function for X_2

Probability for Two Discrete Random Variables

■ Conditional Probability Mass Function

The conditional probability mass function for two random variables X_1 and X_2 is given by

$$P_{X_2|X_1}(x_2 \mid x_1) = \frac{P_{X_1X_2}(x_1, x_2)}{P_{X_1}(x_1)}$$

where $P_{X_2|X_1}(x_2 \mid x_1)$ results in the probability of $X_2 = x_2$ given that $X_1 = x_1$.

$P_{X_1}(x_1) =$ marginal mass function for X_1

Probability for Two Discrete Random Variables

■ Marginal Distributions

The marginal mass function for X_2 that is not equal to zero is

$$P_{X_2}(x_2) = \sum_{\text{all } x_1} P_{X_1X_2}(x_1, x_2)$$

The marginal mass function for X_1 that is not equal to zero is

$$P_{X_1}(x_1) = \sum_{\text{all } x_2} P_{X_1X_2}(x_1, x_2)$$

Probability for Two Discrete Random Variables

■ Properties

If X_1 and X_2 are statistically independent (uncorrelated) random variables, then

$$P_{X_1|X_2}(x_1 \mid x_2) = P_{X_1}(x_1)$$

and

$$P_{X_2|X_1}(x_2 \mid x_1) = P_{X_2}(x_2)$$

The important relationship can be obtained:

$$P_{X_1,X_2}(x_1, x_2) = P_{X_1}(x_1)\,P_{X_2}(x_2)$$

Probability for Two Discrete Random Variables

■ Example: Two Discrete RV's

The time to produce a typical engineering drawing, represented by a random variable X_1, and its quality, represented by a random variable X_2, are under consideration. Suppose X_1 can be 70, 80, 90, or 100 hours. The quality of a drawing can be considered to be moderate, good, and excellent, and X_2 can be considered to be 1, 2, and 3, respectively. Suppose that 100 such drawing are evaluated and the information provided the next table is obtained.

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's

X_2 \ X_1	70	80	90	100
1	15	8	3	2
2	3	4	6	12
3	5	8	12	22

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's
  1. Find the joint PMF of X_1 and X_2.
  2. Plot the marginal PMF of X_1 and X_2.
  3. If only excellent quality drawings are acceptable (i.e., X_2 = 3), plot the conditional PMF of X_2.

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's
  1. The joint PMF $P_{X_1,X_2}(x_1,x_2)$ of X_1 and X_2.

X_2 \ X_1	70	80	90	100
1	0.15	0.08	0.03	0.02
2	0.03	0.04	0.06	0.12
3	0.05	0.08	0.12	0.22

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's
  2. The marginal PMF of X_1 $P_{X_1}(x_1) = \sum_{\text{all } x_2} P_{X_1X_2}(x_1, x_2)$

X_2 \ X_1	70	80	90	100
1	0.15	0.08	0.03	0.02
2	0.03	0.04	0.06	0.12
3	0.05	0.08	0.12	0.22

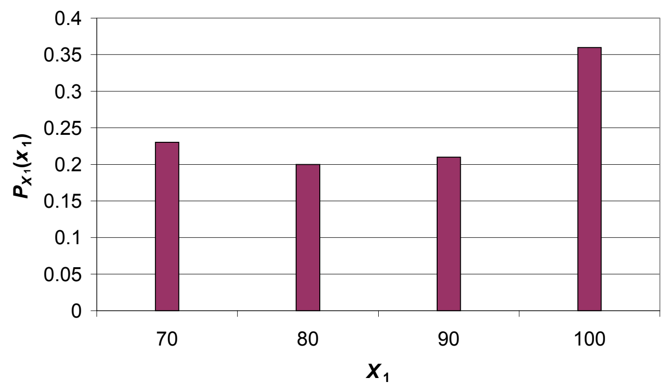
$$P_{X_1}(70) = 0.15 + 0.03 + 0.05 = 0.23$$
$$P_{X_1}(80) = 0.08 + 0.04 + 0.08 = 0.20$$
$$P_{X_1}(90) = 0.03 + 0.06 + 0.12 = 0.21$$
$$P_{X_1}(100) = 0.02 + 0.12 + 0.22 = 0.36$$

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's

$$P_{X_1}(x_1) = \sum_{\text{all } x_2} P_{X_1X_2}(x_1, x_2)$$

Marginal PMF of X1



Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's

The marginal PMF of X_2

$$P_{X_2}(x_2) = \sum_{\text{all } x_1} P_{X_1X_2}(x_1, x_2)$$

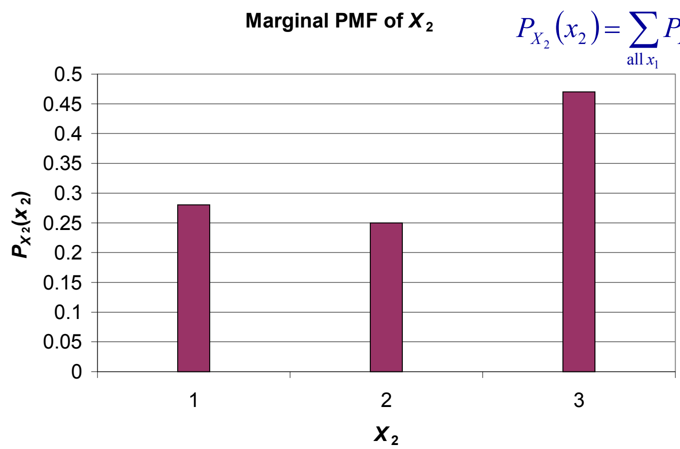
X_2 \ X_1	70	80	90	100
1	0.15	0.08	0.03	0.02
2	0.03	0.04	0.06	0.12
3	0.05	0.08	0.12	0.22

$$P_{X_2}(1) = 0.15 + 0.08 + +0.03 + 0.02 = 0.28$$
$$P_{X_2}(2) = 0.03 + 0.04 + 0.06 + 0.12 = 0.25$$
$$P_{X_2}(3) = 0.05 + 0.08 + 0.12 + 0.22 = 0.47$$

Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's

$$P_{X_2}(x_2) = \sum_{\text{all } x_1} P_{X_1X_2}(x_1, x_2)$$



Probability for Two Discrete Random Variables

- Example (cont'd): Two Discrete RV's
  3. Conditional Probability of X_1 $\quad P_{X_1|X_2}(x_1 | x_2) = \dfrac{P_{X_1X_2}(x_1, x_2)}{P_{X_2}(x_2)}$

$$P_{X_1|X_2}(x_{1_i} | 3) = \frac{P_{X_1X_2}(x_{1_i}, 3)}{P_{X_2}(3)}$$

X_2 \ X_1	70	80	90	100
1	0.15	0.08	0.03	0.02
2	0.03	0.04	0.06	0.12
3	0.05	0.08	0.12	0.22

$$P_{X_1|X_2}(70|3) = \frac{0.05}{0.47} = 0.11$$

$$P_{X_1|X_2}(80|3) = \frac{0.08}{0.47} = 0.17$$

$$P_{X_1|X_2}(90|3) = \frac{0.12}{0.47} = 0.25$$

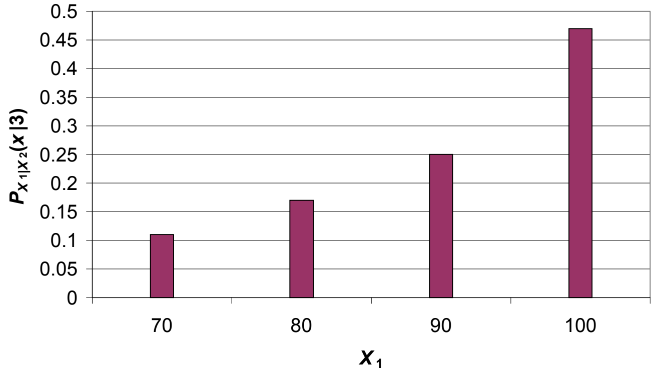
$$P_{X_1|X_2}(100|3) = \frac{0.22}{0.47} = 0.47$$

Probability for Two Discrete Random Variables

- <u>Example (cont'd):</u> Two Discrete RV's

Conditional PMF of *X* | *Y* =3

$$P_{X_1|X_2}\left(x_{1_i}\,|\,3\right)=\frac{P_{X_1X_2}\left(x_{1_i},3\right)}{P_{X_2}(3)}$$



Probability for Continuous Random Vectors

- Joint Probability Density Function (JPDF)

The joint probability density function for a continuous multiple random variable or random vector $\boldsymbol{X} = (X_1, X_2,..,X_n)$ is used to define

$$\mathrm{P}\left(x^l \le X \le x^u\right)=\int_{x_1^l}^{x_1^u}\int_{x_2^l}^{x_2^u}\ldots\int_{x_n^l}^{x_n^u} f_X(x)\,dx_1 dx_2 \ldots dx_n$$

Note that

$$\mathrm{P}(-\infty < X < +\infty)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} f_X(x)\,dx_1 dx_2 \ldots dx_n = 1$$

Probability for Continuous Random Vectors

- Joint Cumulative Distribution Function (JCDF)

  The joint cumulative distribution function of a continuous random variable is defined by

$$F_X(x) = \mathrm{P}(X \le x) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\ldots\int_{-\infty}^{x_n} f_X(x)\,dx_1 dx_2 \ldots dx_n$$

Probability for Continuous Random Vectors

- Properties of JCDF

1. $F_X(\text{all } x \to \infty) = 0$
2. $F_X(x_1, x_2, \ldots, x_i \to -\infty, \ldots, x_n) = 0$, for any $i = 1, 2, \ldots, n$
3. $F_X(x_1, x_2, \ldots, x_i \to -\infty, \ldots, x_k \to -\infty, \ldots, x_n) = 0$, for any values of $x_i, \ldots, x_k$
4. $F_X(x_1, x_2, \ldots, x_i \to +\infty, \ldots, x_n) = F_{Xj}(x_j\colon j = 1, 2, \ldots, n$ and $j \ne i)$, called the marginal distribution of all the random variables except X_i
5. $F_X(x_1, x_2, \ldots, x_i \to +\infty, \ldots, x_k \to +\infty, \ldots, x_n) = F_{Xj}(x_j\colon j = 1, 2, \ldots, n$ and $j \ne i$ to $k)$, called the marginal distribution of all the random variables except X_i to X_k
6. $F_X(\text{all } x \to +\infty) = 1$
7. $F_X(x)$ is a nonnegative and nondecreasing function of x

Probability for Continuous Random Vectors

– The joint density function can be obtained from the a given joint cumulative distribution function as follows:

$$f_X(x) = \frac{\partial^n F_X(x)}{\partial X}$$

That is

$$f_{X_1X_2 \ldots X_n}(x_1, x_2, \ldots, x_3)\frac{\partial^n F_{X_1X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)}{\partial X_1 \partial X_2 \ldots \partial X_n}$$

Probability for Two Continuous Random Variables

- For simplicity, the presentation of the materials in the remaining part of this section is limited to *two* random variables.
- The presented concepts can be generalized to *n* random variables

Probability for Two Continuous Random Variables

- Conditional Probability Density Function

The conditional probability density function for two random variables X_1 and X_2 is given by

$$f_{X_1|X_2}(x_1 \mid x_2) = \frac{f_{x_1,x_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

where $f_{x_1,x_2}(x_1, x_2)$ = joint density function of X_1 and X_2.

$f_{X_2}(x_2)$ = marginal density function for X_2 that is not equal to zero.

Probability for Two Continuous Random Variables

- Conditional Probability Density Function

The conditional probability density function for two random variables X_1 and X_2 is given by

$$f_{X_2|X_1}(x_2 \mid x_1) = \frac{f_{x_1,x_2}(x_1, x_2)}{f_{X_1}(x_1)}$$

where $f_{x_1,x_2}(x_1, x_2)$ = joint density function of X_1 and X_2.

$f_{X_1}(x_1)$ = marginal density function for X_1 that is not equal to zero.

Probability for Two Continuous Random Variables

- Marginal Distributions

The marginal density function for X_2 that is not equal to zero is

$$f_{X_2}(x_2) = \int_{-\infty}^{+\infty} f_{X_1 X_2}(x_1, x_2)\,dx_1$$

The marginal mass function for X_1 that is not equal to zero is

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} f_{X_1 X_2}(x_1, x_2)\,dx_2$$

Probability for Two Continuous Random Variables

- Properties

If X_1 and X_2 are statistically independent (uncorrelated) random variables, then

$$f_{X_1|X_2}(x_1 \mid x_2) = f_{X_1}(x_1)$$

and

$$f_{X_2|X_1}(x_2 \mid x_1) = f_{X_2}(x_2)$$

The important relationship can be obtained:

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$

Probability for Two Continuous Random Variables

- Example: Two Continuous RV's

The joint density functions of two random variables *X* and *Y* can be expressed as

$$f_{X,Y}(x,y)=\begin{cases} c\left(x^2-4\right)\left(y^2-9\right) & \text{for } 0\le x\le 2 \text{ and } 0\le y\le 3 \\ 0 & \text{elsewhere}\end{cases}$$

(a) Determine the constant *c*.
(b) Determine the marginal density function for *X*.
(c) Determine the marginal density function for *Y*.
(d) Are *X* and *Y* statistically independent?
(e) Determine the probability of the following event: $F_{X,Y}(1, 3)$

Probability for Two Continuous Random Variables

- Example (cont'd): Two Continuous RV's

(a)
$$\int_0^3\int_0^2 c\left(x^2-4\right)\left(y^2-9\right)dx\,dy=1$$

or

$$\int_0^3 c\left(y^2-9\right)\left[\frac{x^3}{3}-4x\right]_0^2 dy=\int_0^3 -\frac{16}{3}c\left(y^2-9\right)dy=1.0$$

or

$$-\frac{16}{3}c\left[\frac{y^3}{3}-9y\right]_0^3=1.0$$

or

$$c=\frac{1}{96}$$

Probability for Two Continuous Random Variables

- Example (cont'd): Two Continuous RV's

(a) $$\int_0^3\int_0^2 c(x^2-4)(y^2-9)\,dx\,dy = 1$$

or

$$\int_0^3 c(y^2-9)\left[\frac{x^3}{3}-4x\right]_0^2 dy = \int_0^3 -\frac{16}{3}c(y^2-9)\,dy = 1.0$$

or

$$-\frac{16}{3}c\left[\frac{y^3}{3}-9y\right]_0^3 = 1.0$$

or

$$c = \frac{1}{96}$$

Probability for Two Continuous Random Variables

- Example (cont'd): Two Continuous RV's

(b) $$f_X(x) = \int_0^3 \frac{1}{96}(x^2-4)(y^2-9)\,dy = -\frac{3}{16}(x^2-4)$$

(c) $$f_Y(y) = \int_0^2 \frac{1}{96}(x^2-4)(y^2-9)\,dx = -\frac{1}{18}(y^2-9)$$

(d) $$f_X(x)f_Y(y) = \left[-\frac{3}{16}(x^2-4)\right]\left[-\frac{1}{18}(y^2-9)\right]$$

$$= \frac{1}{96}(x^2-4)(y^2-9) = f_X(x)f_Y(y)$$

$\therefore$ X and Y are statistically independent random variables.

Probability for Two Continuous Random Variables

- Example (cont'd): Two Continuous RV's

  (e)

$$F_{X,Y}(1,3) = \frac{1}{96}\int_0^1 \left(x^2 - 4\right)dx \int_0^3 \left(y^2 - 9\right)dy = 0.6875$$