CHAPTER

Probability, Statistics, and Reliability for Engineers and Scientists

Second Edition

PROBABILITY DISTRIBUTION FOR CONTINUOUS RANDOM VARIABLES

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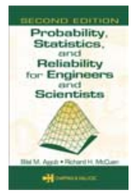

Probability and Statistics for Civil Engineers

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Simulation and Probability Distribution

- Need for Simulation
  - Estimating the probability of failure for both explicit or implicit limit state functions without knowing analytical techniques such as the FORM method.
  - Simulation provides an unique opportunity to understand several important elements related to probability distributions and probabilistic analysis.

Simulation and Probability Distribution

- Need for Simulation (cont'd)
  - Simulation is used to verify the accuracy of structural reliability methods with little background in probability and statistics.
  - Measured data are often very limited, and making decision with small sample sizes increases the risk of incorrect decision.

Simulation and Probability Distribution

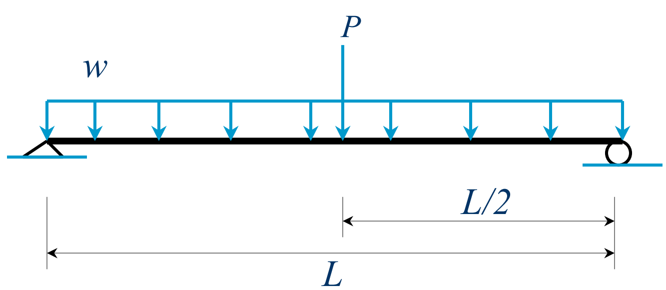
- Monte Carlo Simulation
  - Monte Carlo simulation has six essential elements:
    1. Defining the problem in terms of all the random variables,
    2. Quantifying the probabilistic characteristics of all the random variables (i.e., mean, *COV*, distribution type),
    3. Generating the values of these random variables

Simulation and Probability Distribution

- Monte Carlo Simulation (cont'd)
  4. Evaluating the problem deterministically for each set of all the random variables,
  5. Extracting probabilistic information from *n* observations.
  6. Determining the accuracy and efficiency of the simulation.

Simulation and Probability Distribution

- Formulation of the Problem
  – Consider a simply supported beam as shown



Simulation and Probability Distribution

– Assume both *w* and *P* are random variables.
– Thus, the design bending moment *M* at the midspan of the beam is also a random variable.
– The task now is to evaluate the probabilistic characteristics of the design bending moment using simulation.

Simulation and Probability Distribution

– If the span of the beam is 30 feet, the expression for the design moment can written as

$$M = \frac{wL^2}{8} + \frac{PL}{4}$$
$$= 112.5w + 7.5P$$

– *W* and *P* in this case are called basic random variables.

Simulation and Probability Distribution

- Generation of Random Variables
  - Computer software packages are available (e.g., Excel, Quattro Pro, etc.)
  - The generated random numbers from these packages are called *pseudo random numbers*
  - These numbers are generated from a well-defined and predictable process

Simulation and Probability Distribution

- Midsquare Method
  - This method illustrates the problems associated with deterministic procedures
  - The general procedure is as follows:
    1. Select at random a four-digit number (seed)
    2. Square the number and write the square as an eight-digit number using preceding (lead) zeros if necessary
    3. Use the four digits in the middle as the new random number.
    4. Repeat steps 2 and 3 to generate as many numbers as necessary

Simulation and Probability Distribution

- Example 1: Midsquare Method
  - Consider the seed number 2189. This value would produce the following:

04791721
62678889
46076944
00591361
34963569
92833225
69422224

Simulation and Probability Distribution

- Transformation of Uniform Random Numbers
  - The Uniform Distribution

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

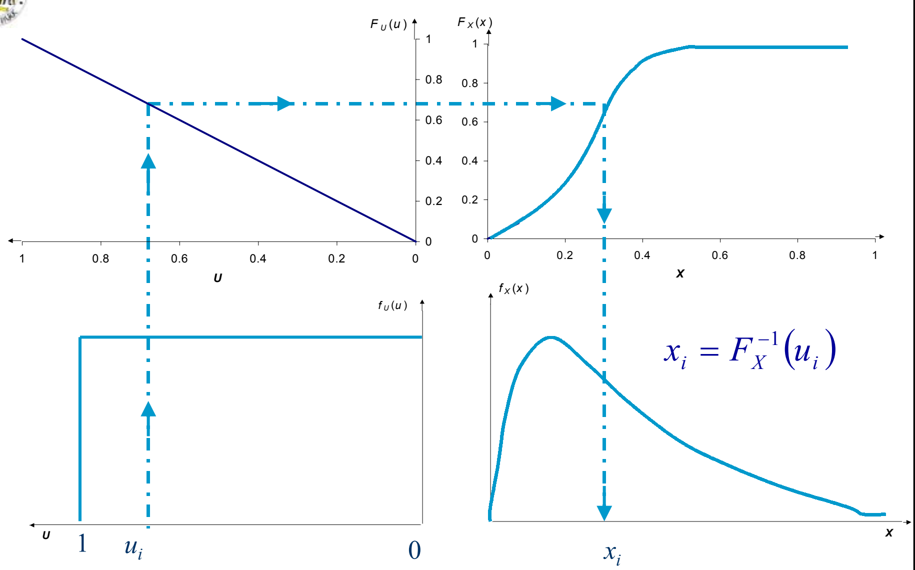
where $a < b$. The mean and variance are given by

$$\mu_X = \frac{a+b}{2} \quad \text{and} \quad \sigma_X^2 = \frac{(b-a)^2}{12}$$

Simulation and Probability Distribution

- Inverse Transformation Technique or Inverse CDF Method
  - In the in inverse transformation technique or inverse CDF method, the CDF of the random variable is equated to the generated random number u_i, that is , $F_X(x_i) = u_i$, and the equation can be solved for x_i as follows:

$$x_i = F_X^{-1}(u_i)$$



Simulation and Probability Distribution

- Example: Normal Distribution

If X is normally distributed, that is $X \sim N(\mu,\sigma^2)$, then $Z = (X - \mu_X)/\sigma_X$ is a standard normal variate, that is, $Z \sim N(0,1)$. It can be shown that

$$u_i = F_X(x_i) = \Phi(z_i) = \Phi\left(\frac{x_i - \mu_X}{\sigma_X}\right)$$

or

$$z_i = \frac{x_i - \mu_X}{\sigma_X}$$

Thus,

$$x_i = \mu_X + \sigma_X z_i = \mu_X + \sigma_X \Phi^{-1}(u_i)$$

Simulation and Probability Distribution

- Example: Lognormal Distribution

If X is lognormally distributed, that is $X \sim LN(\mu,\sigma^2)$, then $Z = (\ln X - \mu_Y)/\sigma_Y$ is a standard normal variate, that is, $Z \sim N(0,1)$. It can be shown that

$$u_i = F_X(x_i) = \Phi(z_i) = \Phi\left(\frac{\ln x_i - \mu_Y}{\sigma_Y}\right)$$

or

$$\ln(x_i) = \mu_Y + \sigma_Y \Phi^{-1}(u_i)$$

Thus,

$$x_i = e^{\left[\mu_Y + \sigma_Y \Phi^{-1}(u_i)\right]}$$

Simulation and Probability Distribution

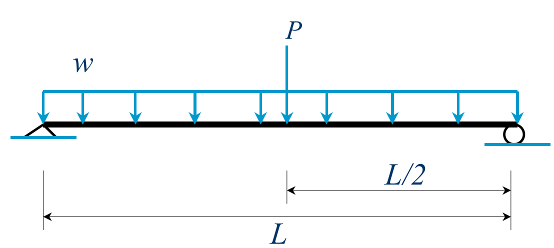
- Example: Simply Supported Beam

  The simply supported beam is subjected to the external loading *w* and *P* as shown in the figure. The probabilistic characteristics of the basic random variables are as follows:

Random Variable	Mean	COV	Standard Deviation	Distribution Type
L	30	-	-	Deterministic
w	2	0.10	0.2	Normal
P	20	0.15	3.0	Lognormal

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam



$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  1. Simulate the design moment *M* for 10 values.
  2. Also, Find the mean, variance, standard deviation, and coefficient of variation of *M* using the simulated sample values.

$$M = \frac{wL^2}{8} + \frac{PL}{4} = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam

Mean (*w*) =	2
Stdev (*w*) =	0.2

Mean (*P*) =	20
Stdev (*P*) =	3

u_1	u_2	*w*	*P*	*M*
0.388248947	0.874573	1.94322	23.47394542	394.6671
0.082540402	0.840615	1.72236	22.95014085	365.8919
0.891083258	0.540006	2.24646	20.07731226	403.3068
0.607604281	0.492971	2.05462	19.72681305	379.0954
0.682506093	0.103666	2.09494	16.38747866	358.5872
0.316169559	0.312569	1.90431	18.38852828	352.1491
0.949955696	0.726221	2.32889	21.63514737	424.2632
0.430819593	0.290549	1.96514	18.21599244	357.6985
0.697860999	0.904197	2.10365	24.0322073	416.9024
0.331143892	0.375488	1.91265	18.8642452	356.6548

Mean (*M*) =	380.9	kip-ft
Variance (*M*) =	727.5	kip-ft^2
Stdev (*M*) =	27.0	kip- ft
COV (*M*) =	0.071	

$$M = 112.5w + 7.5P$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  - Sample Calculations

Consider the second row in the table:

a) w: is normal

$$u_1 = F_W(w) = \Phi(z) = \Phi\left(\frac{w-\mu_W}{\sigma_W}\right)$$

or $$z = \frac{x-\mu_W}{\sigma_W}$$

Therefore,

$$w = \mu_W + \sigma_W z = \mu_W + \sigma_W \Phi^{-1}(u_1)$$
$$= 2 + 0.2\times\Phi^{-1}(0.08254) = 2 + 0.2\times -\Phi^{-1}(1-0.08254)$$
$$= 2 - 0.2\times\Phi^{-1}(0.91746) = 2 - 0.2(1.39) \approx 1.722$$

Simulation and Probability Distribution

- Sample Calculations

B) P: is lognormal $\sigma_Y = \sqrt{\ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right]} = \sqrt{\ln\left[1+\left(\frac{3}{20}\right)^2\right]} = 0.149$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_X^2 = \ln(20) - \frac{1}{2}(0.149)^2 = 2.9846$$

$$u_2 = \Phi\left(\frac{\ln P - \mu_Y}{\sigma_Y}\right) \quad \text{or} \quad \ln P = \mu_Y + \sigma_Y\Phi^{-1}(u_2)$$

or

$$P = e^{\left[\mu_Y + \sigma_Y\Phi^{-1}(u_2)\right]} = e^{\left[2.9846+0.149\times\Phi^{-1}(0.84061)\right]}$$
$$= e^{\left[2.9846+0.149\times\Phi^{-1}(0.84061)\right]} = e^{\left[2.9846+0.149\times 1\right]} = 22.957$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam
  - Sample Calculations
    - Consider the the second row in the table

C) M:

$$M = 112.5w + 7.5P$$
$$= 112.5(1.722) + 7.5(22.957)$$
$$= 365.90 \text{ kip - ft}$$

Simulation and Probability Distribution

- Example (cont'd): Simply Supported Beam

Mean (w) =	2
Stdev (w) =	0.2

Mean (P) =	20
Stdev (P) =	3

u_1	u_2	w	P	M
0.388248947	0.874573	1.94322	23.47394542	394.6671
0.082540402	0.840615	1.72236	22.95014085	365.8919
0.891083258	0.540006	2.24646	20.07731226	403.3068
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$$M = 112.5w + 7.5P$$

Mean (M) =	380.9	kip-ft
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Stdev (M) =	27.0	kip- ft
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