CHAPTER

Probability, Statistics, and Reliability for Engineers and Scientists

Second Edition

PROBABILITY DISTRIBUTION FOR CONTINUOUS RANDOM VARIABLES

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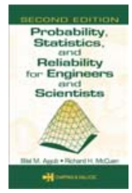

Probability and Statistics for Civil Engineers

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Common Continuous Probability Distributions

- Example: Modulus of Elasticity

  The randomness in the modulus of elasticity (or Young's modulus) *E* can be described by a normal random variable. If the mean and standard deviation were estimated to be 29,567 ksi and 1,507 ksi, respectively,

  1. What is the probability of *E* having a value between 28,000 ksi and 29,500 ksi?
  2. The commonly used Young's modulus *E* for steel is 29,000 ksi. What is the probability of *E* being less than the design value, that is $E \leq$ 29,000 ksi?
  3. What is the probability that *E* is at least 29,000 ksi?
  4. What is the value of *E* corresponding to 10-percentile?

Common Continuous Probability Distributions

■ Example (cont'd): Modulus of Elasticity

$$\mu = 29{,}576 \text{ ksi} \quad \text{and} \quad \sigma = 1{,}507 \text{ ksi}$$

1.

$$P(28{,}000 < E \le 29{,}500) = \Phi\left[\frac{b-\mu}{\sigma}\right] - \Phi\left[\frac{a-\mu}{\sigma}\right]$$

$$= \Phi\left[\frac{29{,}000 - 29{,}576}{1{,}507}\right] - \Phi\left[\frac{28{,}000 - 29{,}576}{1{,}507}\right]$$

$$= \Phi(-0.05) - \Phi(-1.05) = [1 - \Phi(0.05)] - [1 - \Phi(1.05)]$$

$$= (1 - 0.51994) - (1 - 0.85314) = 0.33320$$

2.

$$P(E \le 29{,}000) = \Phi\left(\frac{29{,}000 - 29576}{1{,}507}\right) = \Phi(-0.38)$$

$$= 1 - \Phi(0.38) = 1 - 0.64803 = 0.35197$$

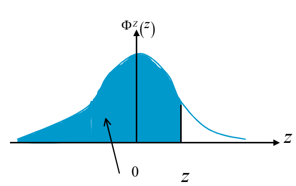
Common Continuous Probability Distributions

■ Sample Table of Standard Normal

$\mu = 0$

$\sigma = 1$

z	$\Phi(z)$	*z*	$\Phi(z)$	*z*	$\Phi(z)$	*z*	$\Phi(z)$
0	0.5	0.2	0.57926	1	0.841345	1.2	0.88493
0.01	0.503989	0.21	0.583166	1.01	0.843752	1.21	0.88686
0.02	0.507978	0.22	0.587064	1.02	0.846136	1.22	0.888767
0.03	0.511967	0.23	0.590954	1.03	0.848495	1.23	0.890651
0.04	0.515953	0.24	0.594835	1.04	0.85083	1.24	0.892512
0.05	0.519939	0.25	0.598706	**1.05**	**0.853141**	**1.25**	**0.89435**
0.06	0.523922	0.26	0.602568	1.06	0.855428	1.26	0.896165
0.07	0.527903	0.27	0.60642	1.07	0.85769	1.27	0.897958
0.08	0.531881	0.28	0.610261	1.08	0.859929	1.28	0.899727
0.09	0.535856	0.29	0.614092	1.09	0.862143	1.29	0.901475
0.1	0.539828	0.3	0.617911	1.1	0.864334		
0.11	0.543795	0.31	0.621719	1.11	0.8665		
0.12	0.547758	0.32	0.625516	1.12	0.868643		
0.13	0.551717	0.33	0.6293	1.13	0.870762		
0.14	0.55567	0.34	0.633072	1.14	0.872857		
0.15	0.559618	0.35	0.636831	1.15	0.874928		
0.16	0.563559	0.36	0.640576	1.16	0.876976		
0.17	0.567495	0.37	0.644309	1.17	0.878999		
0.18	0.571424	**0.38**	**0.648027**	1.18	0.881		
0.19	0.575345	0.39	0.651732	1.19	0.882977		



Common Continuous Probability Distributions

- <u>Example (cont'd):</u> Modulus of Elasticity

$$\mu = 29{,}576 \text{ ksi} \quad \text{and} \quad \sigma = 1{,}507 \text{ ksi}$$

3. $$P(E \geq 29{,}000) = 1 - P(E \leq 29{,}000) = 1 - \Phi\left[\frac{E-\mu}{\sigma}\right]$$
$$= 1 - \Phi\left[\frac{29{,}000 - 29{,}576}{1{,}507}\right]$$
$$= 1 - \Phi(-0.38) = 1 - [1 - \Phi(0.38)]$$
$$= 1 - [1 - 0.64803] = 0.64803$$

4. $$\Phi\left(\frac{E - 29{,}576}{1{,}507}\right) = 0.10 \quad \text{or} \quad \left(\frac{E - 29{,}576}{1{,}507}\right) = \Phi^{-1}(0.10) = -\Phi^{-1}(0.90) = -1.28$$
$$\therefore E = 29{,}576 - 1.28 \times 1507 = 27{,}647 \text{ ksi}$$

Common Continuous Probability Distributions

- Lognormal Distribution
  - Any random variable X is considered to have a lognormal distribution if $Y = \ln(X)$ has a normal probability distribution, where $\ln(x)$ is the natural logarithm to the base e.
  - In many engineering problems, a random variable cannot have negative values due to the physical aspects of the problem.
  - In this situation, modeling the variable as lognormal is more appropriate.

Common Continuous Probability Distributions

■ Lognormal Distribution

- The probability density function (PDF) for the lognormal distribution of a random variable *X* is given by

$$f_X(x) = \frac{1}{x\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu_Y}{\sigma}\right]^2} \quad \text{for } 0 < x < +\infty$$

It is common to use the notation $X \sim \mathrm{LN}\left(\mu_Y, \sigma_Y^2\right)$.

The notation states that X is lognormally distributed with a parameters μ_Y and variance σ_Y^2.

Common Continuous Probability Distributions

■ Lognormal Distribution

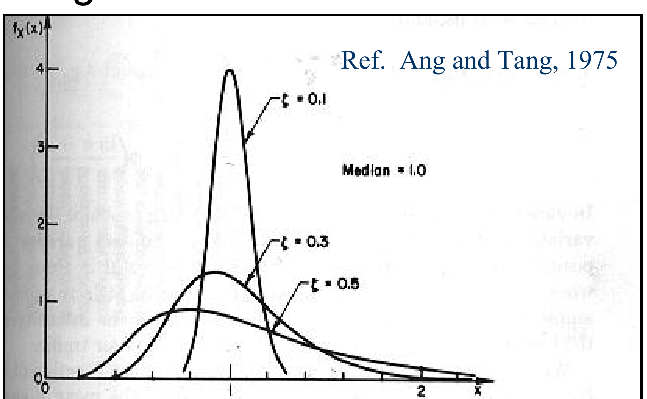


Figure 3.8 Log-normal density functions

Common Continuous Probability Distributions

■ Properties of Lognormal Distribution

1. The values of the random variable *X* are positive
2. $f_X(x)$ is not symmetric density function about the mean value μ_X.
3. The mean value μ_X and σ_X^2 are not equal to the parameters of the distribution μ_Y and σ_Y^2 .
4. They are related as shown in the next viewgraph.
5. In many references, the notations λ_X and ζ_X are used in place of μ_Y and σ_Y^2 , respectively.

Common Continuous Probability Distributions

■ Lognormal Distribution

- Relationships between μ_X, μ_Y, σ_X^2 , and σ_Y^2

$$\sigma_Y^2 = \ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right] \quad \text{and} \quad \mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2$$

These two relations can be inverted as follows:

$$\mu_X = e^{\left(\mu_Y + \frac{1}{2}\sigma_Y^2\right)} \quad \text{and} \quad \sigma_X^2 = \mu_X^2\left(e^{\sigma_Y^2} - 1\right)$$

Note: for small *COV* or $\delta_X = \sigma_X / \mu_X < 0.3$, $\sigma_Y \approx \delta_X$

Common Continuous Probability Distributions

- Useful Properties of Lognormal Distribution
  1. The multiplication of *n* lognormally distributed random variables $X_1, X_2, \ldots, X_n$ is a lognormal distribution with the following statistical characteristics:

$$W = X_1 X_2 X_3 \ldots X_n$$

The mean of *W* is

$$\mu_W = \mu_{Y_1} + \mu_{Y_2} + \mu_{Y_3} + \ldots + \mu_{Y_n}$$

Common Continuous Probability Distributions

The variance or second moment of *W* is

$$\sigma_W^2 = \sigma_{Y_1}^2 + \sigma_{Y_2}^2 + \sigma_{Y_3}^2 + \ldots + \sigma_{Y_n}^2$$

2. Central limit theorem: The multiplication of a number of individual random variables approaches a lognormal distribution as the number of the random variables approaches infinity. The result is valid regardless of the underlying distribution types of the random variables.

Common Continuous Probability Distributions

- Transformation to Standard Normal Distribution

$$Z = \frac{\ln X - \mu_Y}{\sigma_Y} \qquad P(X \le x) = \int_0^{+\infty} \frac{1}{x\sigma_Y\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)^2\right]} dx$$

Changing the variable,

$$P(X \le x) = \int_{-\infty}^{\frac{\ln x - \mu_Y}{\sigma_Y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)$$

It can be shown that

$$P(a \le X \le b) = F_X(b) - F_X(a) = \Phi\left(\frac{\ln b - \mu_Y}{\sigma_Y}\right) - \Phi\left(\frac{\ln a - \mu_Y}{\sigma_Y}\right)$$

Common Continuous Probability Distributions

- <u>Example:</u> Concrete Strength

A structural engineer of the previous example decided to use a lognormal distribution to model the strength of concrete. The mean and standard deviation are same as before, i.e., 3500 psi and 288.7 psi, respectively. What is the probability that the concrete strength is larger than 3600 psi?

$$\mu = 3500 \text{ psi} \qquad \text{and} \qquad \sigma = 288.7 \text{ psi}$$

Common Continuous Probability Distributions

- <u>Example (cont'd):</u> Concrete Strength

$$\sigma_Y^2 = \ln\left[1+\left(\frac{\sigma_X}{\mu_X}\right)^2\right] = \ln\left[1+\left(\frac{288.7}{3500}\right)^2\right] = 0.00678$$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2 = \ln(3500) - \frac{1}{2}(0.00678) = 8.15713$$

The probability that the strength > 3600 psi :

$$P(X > 3600) = 1 - P(X \le 3600) = 1 - \Phi\left[\frac{\ln x - \mu_Y}{\sigma_Y}\right]$$

$$= 1 - \Phi\left[\frac{\ln(3600) - 8.15713}{\sqrt{0.00678}}\right] = 1 - \Phi(0.3833)$$

$$= 0.3507$$

Common Continuous Probability Distributions

- <u>Example (cont'd):</u> Concrete Strength
  - The answer in this case is slightly different from the corresponding value (0.3645) of the previous example for the normal distribution case.
  - It should be noted that this positive property of the random variable of a lognormal distribution should not be used as the only basis for justifying its use.
  - Statistical bases for selecting probability distribution can be used as will be discussed later in Chapter 5.

Common Continuous Probability Distributions

- <u>Example:</u> Modulus of Elasticity

The randomness in the modulus of elasticity (or Young's modulus) *E* can be described by a normal random variable. If the mean and standard deviation were estimated to be 29,567 ksi and 1,507 ksi, respectively,

1. What is the probability of *E* having a value between 28,000 ksi and 29,500 ksi?
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3. What is the probability that *E* is at least 29,000 ksi?
4. What is the value of *E* corresponding to 10-percentile?

Common Continuous Probability Distributions

- <u>Example (cont'd):</u> Modulus of Elasticity

$\mu_X = 29{,}576 \text{ ksi}$ and $\sigma_X = 1{,}507 \text{ ksi}$

$$\text{COV}(X) \text{ or } \delta_X = \frac{\sigma_X}{\mu_X} = \frac{1{,}507}{29{,}576} = 0.051 \leq 0.3$$

Therefore, $\sigma_Y \approx \delta_X = 0.051$

$$\mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y^2 = \ln(29{,}576) - \frac{(0.051)^2}{2} = 10.293$$

1. $$P(\leq 28{,}000 \leq E \leq 29{,}500) = \Phi\left(\frac{\ln(29{,}500) - 10.293}{0.051}\right) - \left(\frac{\ln(28{,}000) - 10.293}{0.051}\right)$$

$$= \Phi(-0.017) - \Phi(-1.04) = (1 - \Phi(0.017)) - (1 - \Phi(1.04))$$

$$= (1 - 0.50678) - (1 - 0.85083) = 0.34405$$

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

2. The probability of E being less than 29,000 ksi is

$$P(E \le 29{,}000) = \Phi\left(\frac{\ln(29{,}000) - 10.293}{0.051}\right) = \Phi(-0.35) = 1 - \Phi(0.35)$$
$$= 1 - 0.63683 = 0.36317$$

3. The probability of E being at least 29,000 ksi is

$$P(E > 29{,}000) = 1 - P(E \le 29{,}000)$$
$$= 1 - \Phi(-0.35) = 1 - (1 - \Phi(0.35))$$
$$= 1 - 1 + 0.63683 = 0.0.63683$$

Common Continuous Probability Distributions

- Example (cont'd): Modulus of Elasticity

4. For 10 - percentile, the E value will computed as follows:

$$\Phi\left(\frac{\ln(E) - 10.293}{0.051}\right) = 0.10$$

or

$$\left(\frac{\ln(E) - 10.293}{0.051}\right) = \Phi^{-1}(0.10) = -\Phi^{-1}(0.90) = -1.28$$

Thus,

$$\ln E = 10.293 - 1.28(0.051)$$

or

$$E = 27{,}659 \text{ ksi}$$

Common Continuous Probability Distributions

■ Exponential Distribution

- The importance of this distribution comes from its relationship to the Poisson distribution.
- For a given Poisson process, the time *T* between the consecutive occurrence of events has an exponential distribution.
- This distribution is commonly used to model earthquakes.

Common Continuous Probability Distributions

■ Exponential Distribution

- The probability density function (PDF) for the exponential distribution of a random variable *T* is given by

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

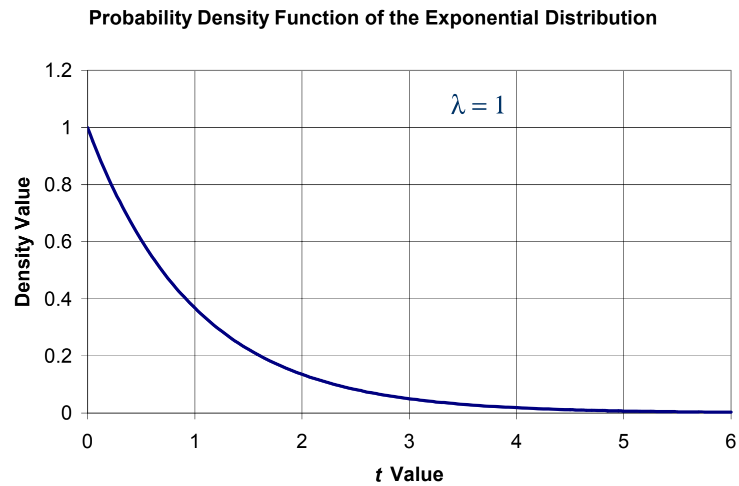
The cumulative distribution function is given by

$$F_T(t) = 1 - e^{-\lambda_t}$$

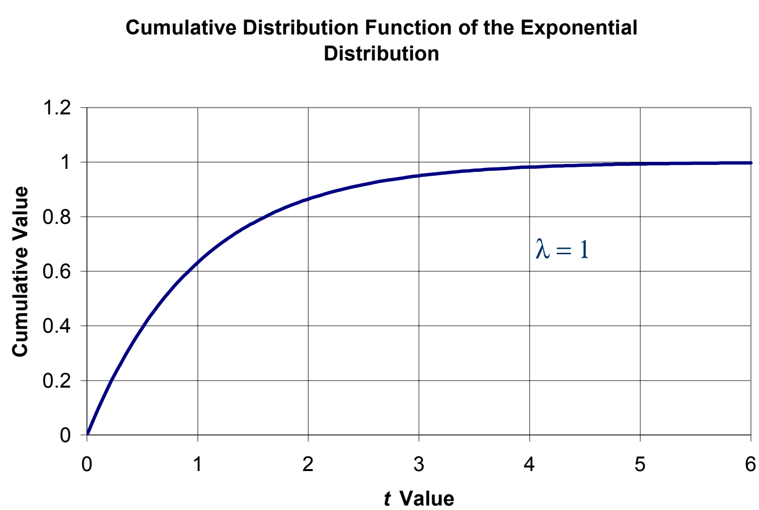
The mean value and the variance are given, respectively, by

$$\mu_T = \frac{1}{\lambda} \quad \text{and} \quad \sigma_T^2 = \frac{1}{\lambda^2}$$

Common Continuous Probability Distributions



Common Continuous Probability Distributions



Common Continuous Probability Distributions

- Exponential Distribution
  - Return Period

    Based on the means of the exponential and Poisson distributions, the mean recurrence time (or return period) is defined as

$$\text{Return Period} = \frac{1}{\lambda}$$

Common Continuous Probability Distributions

- Example: Earthquake Occurrence

  Historical records of earthquake in San Francisco, California, show that during the period 1836 – 1961, there were 16 earthquakes of intensity VI or more. What is the probability that an earthquake will occur within the next 2 years? What is the probability that no earthquake will occur in the next 10 years? What is the return period of an intensity VI earthquake?

Common Continuous Probability Distributions

- Example (cont'd): Earthquake Occurrence

$$\lambda = \frac{\text{Number of Earthquakes}}{\text{Number of Years}} = \frac{16}{1961-1816} = 0.128 \text{ per year}$$

The probability that an earthquake will occur within the next 2 years is

$$\mathrm{P}(T \le 2) = 1 - e^{-\lambda t} = 1 - e^{-(0.128)(2)} = 0.226$$

The probability that no earthquake will occur in the next 10 years is

$$\mathrm{P}(T > 10) = 1 - F_T(10) = 1 - \left(1 - e^{-10\lambda}\right) = e^{-10(0.128)} = 0.278$$

The return period is given by

$$\text{return period} = E(T) = \frac{1}{\lambda} = \frac{1}{0.128} = 7.8 \text{ years}$$