Homework #9 Solution
ENCE 302 - FALL 2001
Due M, 11/19

Problem 1:

Textbook: 5-2

$P_X(x) = \{0.55, 0.33, 0.09, 0.02, 0.01\}$

$$\overline{X} = \sum_{i=1}^{5} xP_X(x) = 0(0.55) + 1(0.33) + 2(0.09) + 3(0.02) + 4(0.01) = 0.61$$

$$S^2 = \sum_{i=1}^{5} \left(x - \overline{X}\right)^2 P_X(x) = (0 - 0.61)^2(0.55) + ... + (4 - 0.61)^2(0.01) = 0.6579$$

$$S = \sqrt{0.6579} = 0.8111$$

Problem 2:

Textbook: 5-3

$$f_X(x) = \frac{3x^2}{26} \qquad \text{for } 1 \le x \le 3$$

$$\mu = \int_1^3 xf_X(x)dx = \int_1^3 x\left(\frac{3x^2}{26}\right)dx = \int_1^3 \frac{3x^3}{26}\,dx = 2.3077$$

$$S^2 = \int_1^3 (x - \mu)^2 f_X(x) = \int_1^3 (x - 2.3077)^2 \left(\frac{3x^2}{26}\right) = 0.2592$$

$$S = \sqrt{0.2592} = 0.5091$$

Problem 3:

Textbook: 5-6

$$\overline{x} = \int_0^b xf_X(x)dx = \int_0^b x\left(\frac{2x}{b^2}\right)dx = \int_0^b \frac{2x^2}{b^2} = \left.\frac{2x^3}{3b^2}\right|_0^b = \frac{2b}{3}$$

$$\therefore b = \frac{3\overline{x}}{2}$$

Problem 4:

Determine the maximum likelihood estimators of the parameters for the normal distribution using an approach similar to the one provided in your notes for the lognormal distribution. If the following five measurements were collected as a sample, estimate the parameters of the normal distribution:

25, 20, 28, 33, 26

***** SOLUTION *****

For normal distribution,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[\frac{x = \mu}{\sigma}\right]^2\right)$$

$$L = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right]$$

The logarithm of this likelihood function is

$$\ln L = \sum_{i=1}^{n} \ln\left\{\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right]\right\} = \sum_{i=1}^{n}\left\{\left[0 - \ln[\sigma\sqrt{2\pi}\right] - \frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right\}$$

$$= \sum_{i=1}^{n}\left\{-[\ln\sigma + \frac{1}{2}\ln 2\pi] - \frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right\}$$

$$= -\sum_{i=1}^{n}\left\{[\ln\sigma + \frac{1}{2}\ln 2\pi] + \frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right\}$$

$$= -\left(n\ln\sigma + \frac{n}{2}\ln\pi\right) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2\sigma^{-2}$$

To maximize the likelihood function, we take the derivatives with respect to u and σ, and set them equal to zero:

$$\frac{\partial(\ln L)}{\partial\mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)(2)(-1) = 0$$

$$\therefore \sum_{i=1}^{n}(x_i - \mu) = 0$$

$$or \quad \sum_{i=1}^{n} x_i - n\mu = 0$$

Hence,

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

In a similar manner

$$\frac{\partial(\ln L)}{\partial\sigma} = -n\frac{1}{\sigma} - \sum_{i=1}^{n}(x_i - \mu)^2(-2)(\sigma^3) = 0$$

$$\therefore \frac{n}{\sigma} = \sum_{i=1}^{n}\frac{(x_i - \mu)^2}{\sigma^3} \Rightarrow n = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{\sigma^2}$$

Hence,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

or

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}$$

So, based on the above results, the maximum likelihood method produced the same results as the method of moments.

25, 20, 28, 33, 26
For the above values,

$$\overline{X} = \frac{25+20+28+33+26}{5} = \frac{132}{5} = 26.4$$

$$S^2 = \frac{(25-26.4)^2 + (20-26.4)^2 + (28-26.4)^2 + (33-26.4)^2 + (26-26.4)^2}{5-1} = 22.3$$

Hence,

$$\left.\begin{aligned} \mu &= \overline{X} = 26.4 \\ \sigma^2 &= S^2 = 22.3 \end{aligned}\right. \quad \text{for normal distribution}$$

<u>Problem 5</u>:

Textbook: 5-10

(a) H_o: $\mu=10$

$$P(\overline{x} > 12) = P\left(Z > \frac{12-10}{5/\sqrt{25}}\right) = P(Z > 2) = 1 - P(Z < 2)$$

$$= 1 - 0.9772 = 0.0228$$

(b) $$P(\overline{x} > 12) = P\left(Z > \frac{12-10}{5/\sqrt{100}}\right) = P(Z > 4) = 1 - P(Z < 4)$$

$$= 1 - 1 = 3.17\text{x}10^{-5}$$

(c) $$P(\overline{x} > 12) = P\left(Z > \frac{12-10}{10/\sqrt{25}}\right) = P(Z > 1) = 1 - P(Z < 1)$$

$$= 1 - 0.8413 = 0.1587$$

(d) $$P(\overline{x} > 6) = P\left(Z < \frac{6-10}{10/\sqrt{25}}\right) = P(Z < -2) = 0.0228$$