Homework #8 Solution
ENCE 302 - FALL 2001
Due M, 11/12

Problem 1:

Textbook: 4-2

	$Y \backslash X$	X = Thermal Test: A	NA	PA	$P_Y(y)$
Y = Magnetic Field	A	0.80	0.02	0.00	0.82
	NA	0.04	0.02	0.02	0.08
	PA	0.04	0.02	0.04	0.10
$P_X(x)$		0.88	0.06	0.06	1.00

Problem 2:

The joint probability density function of two random variables X and Y is expressed as follows:

$$f_{X,Y} = \begin{cases} c\left(x^2 + xy + y^2\right) & \text{for } 0 \le x \le 2 \text{ and } 2 \le y \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Determine the constant c such that $f_{X,Y}(x,y)$ is a legitimate joint density function.
(b) Determine the marginal density function for X.
(c) Determine the marginal density function for Y.
(d) Are X and Y statistically independent (uncorrelated)? Justify your answer.
(e) Determine the probability of the following events:
1. P($Y > 3 \mid X = 1$)
2. $F_{X,Y}(1,3)$

***** SOLUTION *****

(a) Determination of c:

$$\int_2^4 \int_0^2 c\left(x^2 + xy + y^2\right) dx\, dy = 1$$

or $$c\int_2^4 \left(\frac{x^3}{3} + \frac{x^2}{2} y + y^2 x\right)\Bigg|_0^2 dy = c\int_2^4 \left(\frac{8}{3} + 2y + 2y^2\right) dy = c\left(\frac{8}{3} y + 2\frac{y^2}{2} + 2\frac{y^3}{3}\right)\Bigg|_2^4 = 1$$

or $$c = \frac{1}{54.67}$$

(b) Marginal DF for X:

$$f_X(x) = \int_2^4 \frac{1}{54.67}\left(x^2 + xy + y^2\right) dy = \frac{1}{54.67}\left(2x^2 + 6x + \frac{56}{3}\right)$$

(c) Marginal DF for Y:

$$f_Y(y) = \int_0^2 \frac{1}{54.67}\left(x^2 + xy + y^2\right) dx = \frac{1}{54.67}\left(\frac{8}{3} + 2y + 2y^2\right)$$

(d) X and Y are not statistically independent because $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$.

(e) Probabilities:

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x^2 + xy + y^2}{2x^2 + 6x + \frac{56}{3}}$$

1. $P(Y > 3 \mid X = 1) = \dfrac{1}{26.67}\displaystyle\int_3^4 \left(1 + y + y^2\right)dy = 0.63125$

2. $F_{X,Y}(1,3) = \displaystyle\int_2^3\int_0^1 \frac{1}{54.67}\left(x^2 + xy + y^2\right)dxdy = 0.1448$

<u>Problem 3:</u>

Textbook: 4-11

The conditional probability mass function $P_{X|Y}(X = \text{PA})$ for different y values is given as

$$P(\text{PA} \mid \text{A}) = \frac{0}{0.82} = 0$$

$$P(\text{PA} \mid \text{NA}) = \frac{0.02}{0.08} = 0.25$$

$$P(\text{PA} \mid \text{PA}) = \frac{0.04}{0.10} = 0.4$$

<u>Problem 4</u>:

Textbook: 4-20

$$\delta = \frac{PL}{AE}$$

1. The first-order mean

$$u_\delta = \frac{L\mu_P}{A\mu_E}$$

2. The first-order variance:

$$\frac{\partial\delta}{\partial P} = \frac{L}{AE}$$

$$\frac{\partial\delta}{\partial E} = -\frac{LP}{AE^2}$$

(a) P and E are uncorrelated:

$$\sigma_\delta^2 = \sum_{i=1}^{n}\left(\frac{\partial\delta(x)}{\partial X_i}\right)^2 Var(X_i) = \left(\frac{L}{A\mu_E}\right)^2\sigma_P^2 + \left(-\frac{L\mu_P}{A\mu_E^2}\right)^2\sigma_E^2$$

(b) P and E are correlated

$$\sigma_\delta^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial\delta(x)}{\partial X_i}\right)\left(\frac{\partial\delta(x)}{\partial X_j}\right)\text{cov}(X_i, X_j)$$

$$\sigma_\delta^2 = \left(\frac{L}{A\mu_E}\right)^2\sigma_P^2 + \left(-\frac{L\mu_P}{A\mu_E^2}\right)^2\sigma_E^2 + 2\left(\frac{L}{A\mu_E}\right)\left(-\frac{L\mu_P}{A\mu_E^2}\right)\text{cov}(P,E)$$

Problem 5:

The rate of steady water flow per time unit at a constant depth in a prismatic open channel can be estimated using the following Manning formula:

$$Q = \frac{C_m}{n} A R^{2/3} S^{1/2}$$

where C_m is a constant of value = 1 in SI system, n = Manning roughness factor, R = hydraulic radius, and S = slope of the bottom of the channel. Consider a trapezoidal cross section made with gravel. Assume C_m is a constant. However, n, A, R, and S are statistically independent random variables with mean values of 0.029, 8.0 m^2, 1.1 m, and 0.003, respectively, and the corresponding *COV*'s are 0.30, 0.1, 0.1, and 0.1, respectively. No distributional information is available.

(a) Determine the first-order mean and standard deviation of Q.

(b) Determine the second-order mean of Q using the following equation:

$$E(Y) \approx g\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n}\right) + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial^2 g}{\partial X_i^2}\right)\mathrm{Var}(X_1)$$

(c) Compare the results of Parts (a) and (b).

***** SOLUTION *****

(a) First-order mean:

$$E(Q) \approx \frac{1}{0.029}(8)(1.1)^{2/3}(0.003)^{1/2} = 16.1\frac{\mathrm{m}^3}{s}$$

First-order variance:

$$Var(Q) = \left(\frac{1}{0.029^2}(8)(1.1)^{2/3}(0.003)^{1/2}\right)^2(0.029\times 0.3)^2 + \left(\frac{1}{0.029}(8)(1.1)^{2/3}(0.003)^{1/2}\right)^2(8\times 0.1)^2$$

$$+\left(\frac{1}{0.029}(8)\left(\frac{2}{3}\right)(1.1)^{-1/3}(0.003)^{1/2}\right)^2(1.1\times 0.1)^2 + \left(\frac{1}{0.029}(8)(1.1)^{2/3}\left(\frac{1}{2}\right)(0.003)^{1/2}\right)^2(0.003\times 0.1)^2$$

$$= 23.33 + 2.59 + 1.15 + 0.65 = 27.72$$

Therefore, standard deviation $\sigma_Q = 5.265\,\frac{\mathrm{m}^3}{s}$

(b) Second-order mean:

$$E(Q) \approx 16.1 + \frac{1}{2}\left[\left(\frac{(2)(1)}{0.029^2}(8)(1.1)^{2/3}(0.003)^{1/2}\right)(0.029 \times 0.3)^2\right.$$

$$+ (8 \times 0.1)^2 \times 0 + \left(\frac{1}{0.029}(8)\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)(1.1)^{-4/3}(0.003)^{1/2}\right)(1.1 \times 0.1)^2$$

$$+ \left(\frac{1}{0.029}(8)(1.1)^{2/3}\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(0.003)^{-3/2}\right)(0.003 \times 0.1)^2$$

$$= 16.1 + \frac{1}{2}(2.898 + 0 - 0.036 - 0.04) = 17.51\,\frac{\text{m}^3}{s}$$

Comparison of means: $\left|\frac{16.1 - 17.51}{17.51}\right| \times 100 = 8.05\%$ error