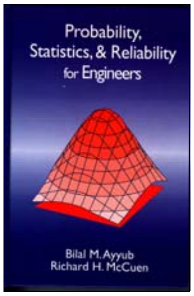
University of Maryland at College Park
Department of Civil and Environmental Engineering
ENCE 302 – Probability and Statistics for Civil Engineers



SOLUTION OF MIDTERM EXAM
Closed Book and Notes (8.5''x11" sheet allowed) for 50 minutes
October 24, 2001

Instructor: Dr. Ibrahim A. Assakkaf

"***Show your work & state all your assumptions***"

Student Name:_________SAMPLE__________

SSN:____123-45-6789_______

Grade:_____100 ☺_____

ENCE 302 MIDTERM EXAM October 24, 2001 Name:______________________

Problem 1 (25 points)

I. True or False (10 points)
If each of the following statements is true, circle T, otherwise circle F:

(1) If events E_1 and E_2 are statistically independent, then T (F)

$\mathrm{P}(E_1 \cup E_2) = \mathrm{P}(E_1) + \mathrm{P}(E_2)$

False, The correct answer is $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1)P(E_2)$

(2) If events E_1 and E_2 are mutually exclusive, then (T) F

$\mathrm{P}(E_1 \mid E_2) = 0$

True: because $P(E_1 \cap E_2) = P(E_1 \mid E_2)P(E_2) = 0 \Rightarrow P(E_1 \mid E_2) = 0$

(3) If E_1, E_2, and E_3 are mutually exclusive and collectively exhaustive, then (T) F

$\mathrm{P}(E_1) = 1 - \mathrm{P}(E_2) - \mathrm{P}(E_3)$

(4) $\mathrm{P}(E_1 \mid E_2) = \dfrac{\mathrm{P}(E_1)}{\mathrm{P}(E_2)}$ T (F)

False, The correct answer is $\mathrm{P}(E_1 \mid E_2) = \dfrac{\mathrm{P}(E_1 \cap E_2)}{\mathrm{P}(E_2)}$

(5) $\mathrm{P}(\overline{E}_1 \mid E_2 \cup E_3) = 1 - \mathrm{P}(E_1 \mid \overline{E_2 \cup E_3})$ T (F)

False, The correct answer is $\mathrm{P}(\overline{E}_1 \mid E_2 \cup E_3) = 1 - \mathrm{P}(E_1 \mid E_2 \cup E_3)$

(6) Probability is a numerical measure of the likelihood of occurrence of an event relative to a set of alternative events. (T) F

(7) Statistics is the probability of successful performance of an engineering system. It is the converse of the term "probability of failure." T (F)

(8) Data can be measured on one of the following scales: (a) ordinal scale; (b) interval scale; (c) nominal scale, and (d) ratio scale. The nominal scale of measurement is the lowest level. (T) F

(9) In the Taylor series expansion, as the separation distance (h) gets smaller the solution diverges from the true value. T (F)

(10) If a pair of dice is rolled simultaneously, the probability that the sum of the dots is an even number is equals 50%. (T) F

II. **Cumulative Mass Function** (15 points)

For the following cumulative mass function, derive the mass function:

$$F_X(x) = \begin{cases} 0.1 & x = 1 \\ 0.3 & x = 2 \\ 0.3. & x = 3 \\ 0.4 \quad for & x = 4 \\ 0.8 & x = 5 \\ 0.9 & x = 6 \\ 1.0 & x = 7 \end{cases}$$

(a) Graph both the probability and cumulative mass functions.

(b) Find the probability that:

i) $P(x = 3)$

ii) $P(3 < x \leq 6)$

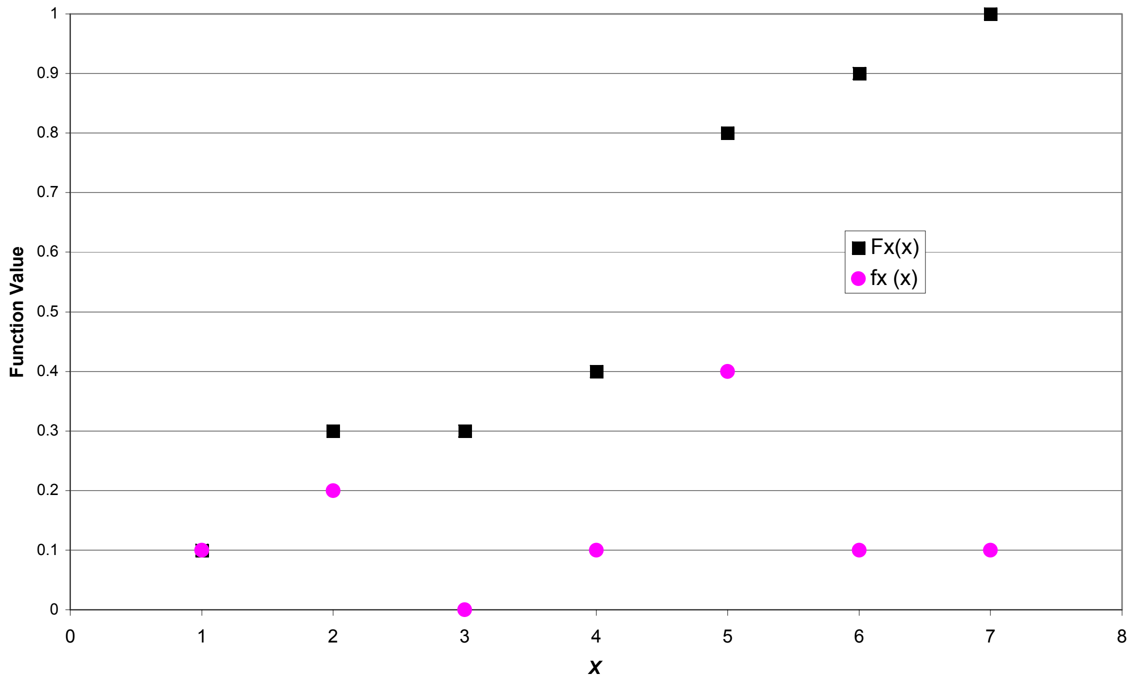
iii) $P(x \geq 6)$

***** SOLUTION *****

(a)

$$f_X(x) = \begin{cases} 0.1 & x = 1 \\ 0.2 & x = 2 \\ 0.0 & x = 3 \\ 0.1 \quad for & x = 4 \\ 0.4 & x = 5 \\ 0.1 & x = 6 \\ 0.1 & x = 7 \end{cases}$$

x	F_x(x)	fx (x)
1	0.1	0.1
2	0.3	0.2
3	0.3	0.0
4	0.4	0.1
5	0.8	0.4
6	0.9	0.1
7	1.0	0.1



(b)

i) $= P(X \le 3) - P(X \le 2) = 0.3 - 0.3 = 0.0$

ii) $= P(X \le 6) - P(X \le 3) = 0.9 - 0.3 = 0.6$

iii) $= 1.0 - P(X < 6) = 1.0 - 0.8 = 0.2$

Problem 2 (25 points)

(a) A house owner is planning to replace the old boiler in his basement. The new boiler has to be delivered by the vendor, but the work can't start on time unless after the arrival of both the electrician and the mechanic. If the probability that the electrician will show up on time is 0.8 and the mechanic is 0.75, and if the probability that the work will start as scheduled is 0.55, what is the probability that the boiler will be delivered on time?

(b) Find the probability that the outcome of a single draw from a card deck is an ace given that five cards, including two aces, have already been drawn and discarded from the deck.

(c) The compressive strength of concrete specimen follows normal distribution with a mean value (μ) of 2.4 ksi and a coefficient of variation of 0.2. If the applied stress is 2.5 ksi, find the probability of failure.

*****SOLUTION *****

(a) Probability that the boiler will be delivered= P(B)
Probability that the electrician will show up on time = P(E)
Probability that the mechanic will show up on time = P(M)
Probability that the work will start as scheduled = P(W)

P(E) = 0.8, P(M) = 0.75, and P(W) = 0.55
Since that the events are independent, the probability that work will start on time is:
$P(W) = P(B \cap E \cap M) = P(B) * P(E) * P(M)$
$0.55 = P(B) * 0.8 * 0.75$,
$P(B) = 0.92$

(b) Number of cards left = 52 –5 = 47
Number of aces left = 4 –2 = 2

$$\text{Probability} = \frac{2}{47} = 0.04255$$

(c) $\mu = 2.4$, $COV = 0.2$, $COV = \sigma / \mu$

$$\sigma = COV\,\mu = 0.2\,(2.4) = 0.48$$

$$\Phi(z) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{2.5-2.4}{0.48}\right) = \Phi(0.208)$$

From Appendix A-1, Φ(0.20) = 0.57926 and Φ(0.21) = 0.583166. Therefore, using linear interpolation:

z	$\Phi(z)$
0.20	0.57926
0.208	$\Phi(z)$
0.21	0.583166

$$\Rightarrow \frac{\Phi(z)-0.57926}{0.583166-0.57926} = \frac{0.208-0.2}{0.21-0.2} \Rightarrow \Phi(z) = 0.5824$$

P(failure) = P(x <2.5) = Φ(0.208) = **0.5824**

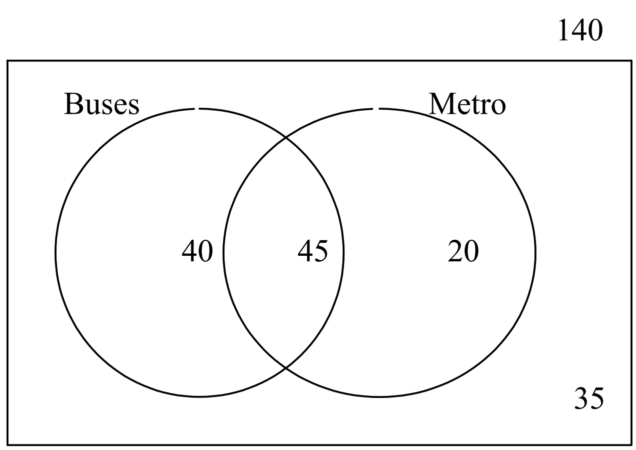
Problem 3 (25 points)
A survey of 140 college students showed that in the last week 45 students used both buses and metro trains for their transportation, 20 students used metro trains but not buses, while 25% of the students didn't use either but drove their private vehicles.

(a) How many students used buses but not metro trains?
(b) How many students used metro trains?
(c) How many students used buses?
(d) Draw a Venn diagram showing this information.
(e) If this survey was used to forecast future transportation trends, what is the probability that a student will only use a bus?

***** SOLUTION *****

(a) Number of students who didn't use either method = 0.25 * 140 = 35 students
Thus, number of students who used buses but not metro trains = 140-45-20-35 = 40 students.

(b) 140-40-35 = 65 students.
(c) 140-20-35 = 85 students.
(d)



(e) Probability = 40 / 140 = 0.286 = 28.6%.

Problem 4 (25 points)

For the following cumulative density function (CDF):

$$F_X(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{x^2}{120{,}000} & \text{for} \quad 0 \le x \le 346.4 \\ 3.0 & \text{for} \quad x > 346.4 \end{cases}$$

(a) Determine the probability density function (PDF) and sketch it.
(b) Determine the mean of X.
(c) Determine the variance of X.
(d) Calculate the standard deviation of X, and
(e) Calculate the coefficient of variation of X.

***** SOLUTION *****

(a) Sketch of PDF:

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right] = \frac{d}{dx}\left[\frac{x^2}{120{,}000}\right] = \frac{2x}{120{,}000} = \boxed{\frac{x}{60{,}000}}$$

when $x = 346.4$, $f_X(x) = \dfrac{346.4}{60{,}000} = \dfrac{1}{173.2}$

(b) Mean of X:

$$\mu = \int_0^{346.4} x f_X(x)dx = \int_0^{246.4} x \frac{x}{60{,}000} = \left.\frac{x^3}{180{,}000}\right|_0^{300} = \frac{(346.4)^3}{180{,}000} = \boxed{230.92}$$

(c) Variance of X:

$$\sigma^2 = \int_0^{346.4} (x-\mu)^2 f_X(x)dx = \int_0^{346.4} (x-\mu)^2 \frac{x}{60{,}000}dx = \int_0^{346.4} \left(x^2 - 2\mu x + \mu^2\right)\frac{x}{60{,}000}dx =$$

$$= \frac{1}{60{,}000}\int_0^{246.4}\left(x^3 - 2\mu x^2 + \mu^2 x\right)dx = \frac{1}{60{,}000}\left[\frac{x^4}{4} - \frac{2\mu x^3}{3} + \frac{\mu^2 x^2}{2}\right]_0^{346.4} = \boxed{6{,}666}$$

(d) Standard Deviation:

$$\sigma = \sqrt{\text{Var}} = \sqrt{6{,}666} = \boxed{81.65}$$

(e) Coefficient of Variation (*COV* or δ) :

$$\text{cov} = \frac{\sigma}{\mu} = \frac{81.65}{230.92} = \boxed{0.35}$$

Rule Type	Operations
Identity Laws	$A \cup \varnothing = A,\ A \cap \varnothing = \varnothing,\ A \cup S = S,\ A \cap S = A$
Idem potent Laws	$A \cup A = A,\ A \cap A = A$
Complement Laws	$A \cup \overline{A} = S,\ A \cap \overline{A} = \varnothing,\ \overline{\overline{A}} = A,\ \overline{S} = \varnothing,\ \overline{\varnothing} = S$
Commutative Laws	$A \cup B = B \cup A,\ A \cap B = B \cap A$
Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
De Morgan's Law	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \overline{(E_1 \cup E_2 ... \cup E_n)} = \overline{E_1} \cap \overline{E_2} ... \cap \overline{E_n}$ $\overline{(A \cap B)} = \overline{A} \cup \overline{B}, \overline{(E_1 \cap E_2 \cap ... \cap E_n)} = \overline{E_1} \cup \overline{E_2} \cup ... \cup \overline{E_n}$
Combinations of Laws	$\overline{(A \cup (B \cap C))} = \overline{A} \cap \overline{(B \cap C)} = (\overline{A} \cap \overline{B}) \cup (\overline{A} \cap \overline{C})$

$$COV = \frac{\text{standard deviation}}{\text{mean (or average)}} = \frac{S}{\overline{X}}$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{X})^2$$

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right]$$

OR

$$S^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\overline{X}^2\right]$$

■ Continuous Random Variables

- If X is a continuous random variable with PDF $f_X(x)$, the following expressions can be used to compute the mean, variance, and skewness:

$$E(X) = \mu = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

$$\text{Variance} = \text{Var}(X) = \sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\text{Skewness} = \lambda = \int_{-\infty}^{+\infty} (x-\mu)^3 f_X(x)\,dx$$

■ Discrete Random Variables

- If X is a discrete random variable with PMF $P_X(x)$, the following expressions can be used to compute the mean, variance, and skewness:

$$E(X) = \mu = \sum_{i=1}^{n} x_i P_X(x_i)$$

$$\text{Variance} = \text{Var}(X) = \sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P_X(x_i)$$

$$\text{Skewness} = \lambda = \sum_{i=1}^{n} (x_i - \mu)^3 P_X(x_i)$$

SECOND DIE

FIRST DIE

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$$f_X(x) = \frac{dF_X(x)}{dx}$$

where $f_X(x)$ = probability density function

and $F_X(x)$ = cumulative distribution function

$$f(x_0 + h) = f(x_0) + hf^1(x_0) + \frac{h^2}{2!} f^2(x_0) + \frac{h^3}{3!} f^3(x_0) + ... + \frac{h^n}{n!} f^n(x_0)$$